MIR@W

MATHEMATICAL INTERDISCIPLINARY RESEARCH

PROGRAMME SUMMER TERM 1998/9

Monday 7th June 1999

TUXEDO The UK Spatially Extended Dynamics Organisation

Organisers: David Rand

10.00 - 11.00 Coffee

11.30 - 12.10 Daniel Margerit (Warwick): Singular perturbation equations for 3D excitable media.

Abstract: *Excitable media, such as nerve fibers and heart tissue, are typically modeled with reaction diffusion equations containing two chemical species that evolve on very different time scales. In three dimensions solutions to these equations take the form of rotating scroll waves (interfaces) ending on filaments. The ratio of the two times scale defines a natural small parameter epsilon. Exploiting the inherent smallness of epsilon, singular perturbation methods are used to derive three-dimensional equations for each of two boundary layers: interface region (scroll) and filament region (core), and for the associated outer region. This provides the first fully three-dimensional description of the wavefronts and filaments in excitable media.*

12.20 - 1.00 Nigel Burroughs: T-cell self assessment: physiologically structured growth models.

Abstract. *Growth models incorporating soluble growth factors will be discussed, incorporating receptor density modulation. Such models show robust growth characteristics (growth is only weakly dependent on the specificity of the T-cell clone) while producing a high level of competition between clones. Thus the optimal T-cell clone present is selected.*

2.00 - 2.40 Hans Henrik Rugh (Warwick): Coupled maps and analytic function spaces.

Abstract: *We study small deformations of an infinite direct product of uniformly expanding circle maps (all analytic). We show that there is a natural invariant measure with holomorphic marginal densities and that time correlations decay exponentially. In the proofs we introduce new techniques of renormalization mappings to obtain uniform bounds for a Perron Frobenius operator associated with the coupled map.*

2.50 - 3.10 Vesna Kadelburg (Cambridge): Explicit Bounds for continuation of breathers.

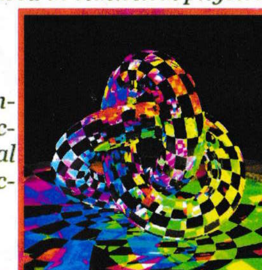
Abstract: *We find explicit estimates for the bounds for continuation of descrete breathers in one-dimensional chains, for some particular cases of the external potential. The estimates are based on the existance proof of MacKay and Aubry, by continuation from the anti-integrable (uncoupled) limit.*

3.40 - 4.20 Guy Gielis (Cambridge): A new phase transition?

Abstract: *Starting from a discrete time stochastic dynamics on* $\{0,1\}^{\mathbb{Z}^d}$, *we define a deterministic map on* $[0,1]^{\mathbb{Z}^d}$. *The almost trivial connection with the intensively studied underlying stochastic system provides a great deal of insight in these new maps. Exploiting this, we translate a notion of phase transition generally used in statistical physics to the context of (infinitely dimensional) deterministic dynamics. This is joint work with Robert MacKay.*

4.30 - 5.10 Tom Bridges (Surrey): Symplectic Pattern Formation.

Abstract: *Symplectic pattern formation is the study of patterns in spatially extended systems which also have some element of conservation or symplecticity. The natural structure for such systems is multi-symplecticity, where distinct symplectic structures are assigned for the space and time directions. In this talk we will focus on the transverse instability of solitary waves and fronts. Given a solitary wave or front propagating in one spatial direction, a transverse instability is an instability propagating in a direction transverse to the original wave direction. We show that multi-symplecticity gives a natural geometric instability criterion for such patterns.*



5.10 Wine and snacks in the Mathematics Institute Common Room.

All programmes will take place in Lecture Room MI 1 of the Mathematics Institute

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http://www.maths.warwick.ac.uk/

graphics courtesy of Bob Scharein - http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html

Singular perturbation equations for 3D excitable media

Daniel Margerit and Dwight Barkley

http://www.maths.warwick.ac.uk/~dmargeri/

Waves in 2-d and 3-d excitable media

Reaction-diffusion model

Excitable media, such as nerve fibers and heart tissue, are typically modeled with reaction diffusion equations containing two chemical species that evolve on very different time scales :

$$\boxed{\begin{aligned} \epsilon\partial u/\partial t &= \epsilon^2\Delta^2 u + f(u,v) \\ \partial v/\partial t &= \epsilon\delta\Delta^2 v + g(u,v) \end{aligned}}$$

with $\epsilon \ll 1$.

In the Dwight model

$$\begin{aligned} f(u,v) &= u\,(1-u)\,(u-u_{th}) \\ g(u,v) &= u-v \\ u_{th} &= (v+b)/a \end{aligned}$$

Waves solutions

This equation has stationnary waves solutions :

- Spiral waves in 2-d :

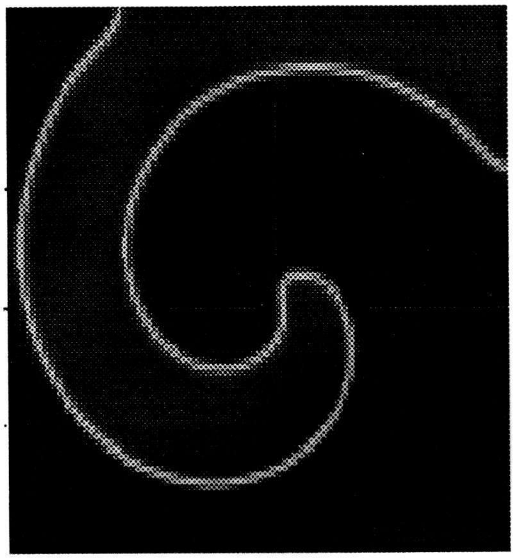
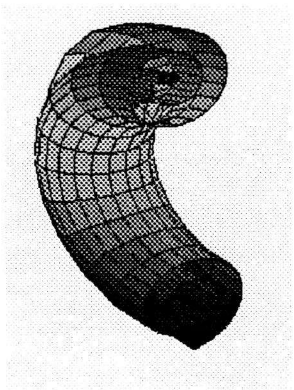
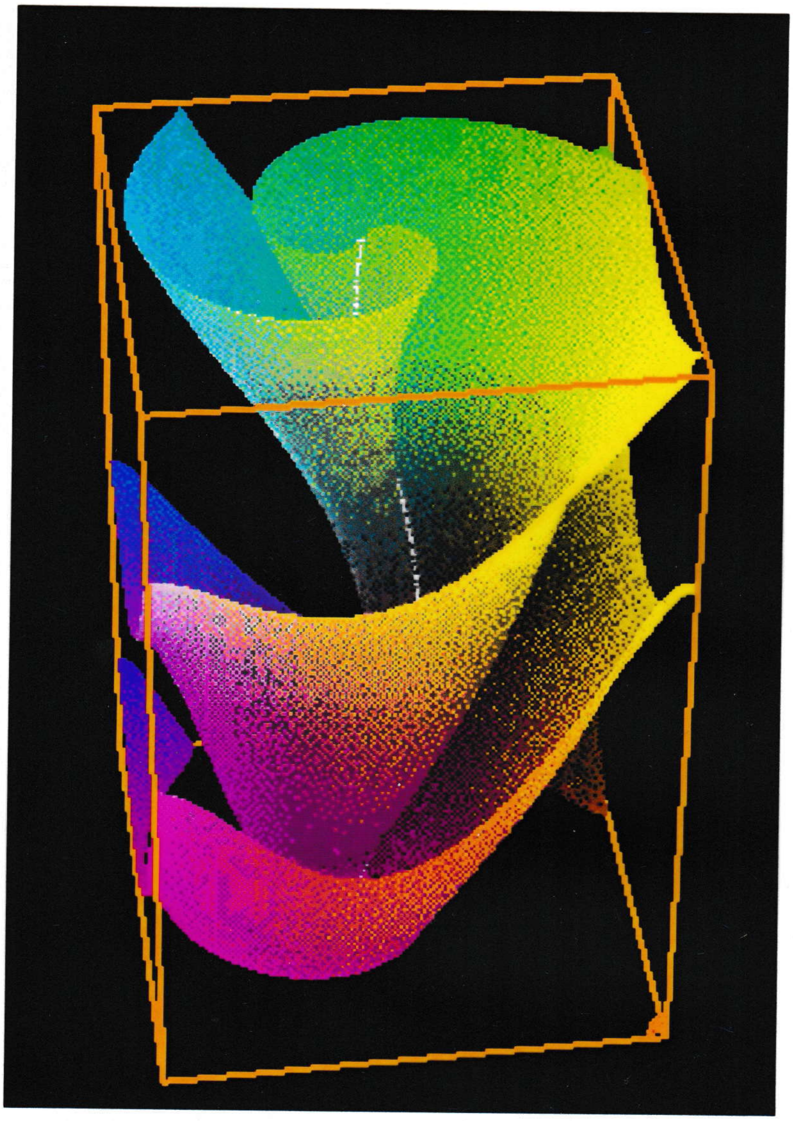


Figure 1: The u field

- Scroll waves in 3-d :



Reasons for the motion of the filament ?



Twist τ _ Scroll Filament.

Coordinates and geometry

Filament

$$\mathbf{X} = \mathbf{X}(s,t) \qquad T : \text{torsion}$$

$$K : \text{curvature} \qquad (\mathbf{t}, \mathbf{n}, \mathbf{b})$$

Local coordinates

$$\mathbf{M}(r, \varphi, s) \qquad (\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{t})$$

$$\mathbf{x} = \mathbf{OM} = \mathbf{X}(s,t) + r\mathbf{e}_r(\varphi, s, t)$$

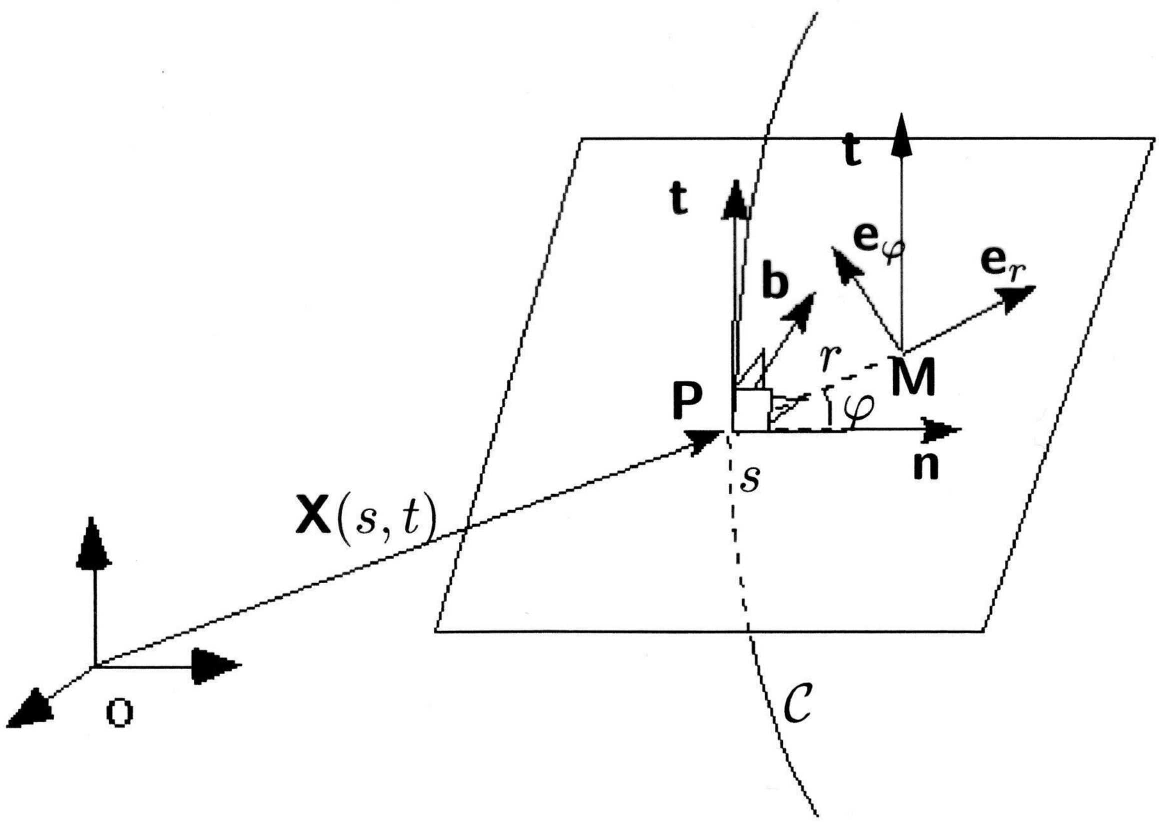


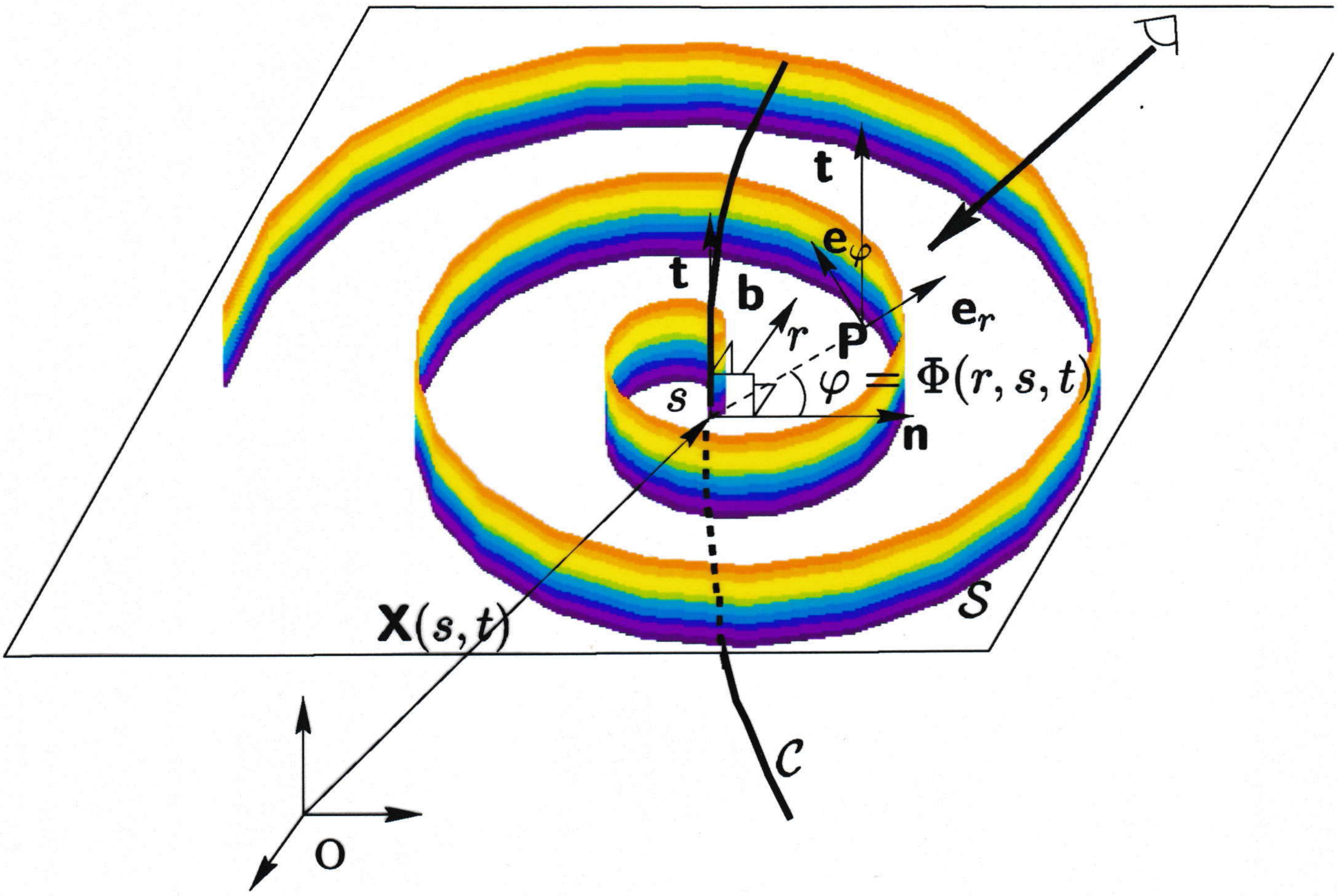
Figure 2: The central curve $\mathcal{C}$ and the local co-ordinates of the scroll filament

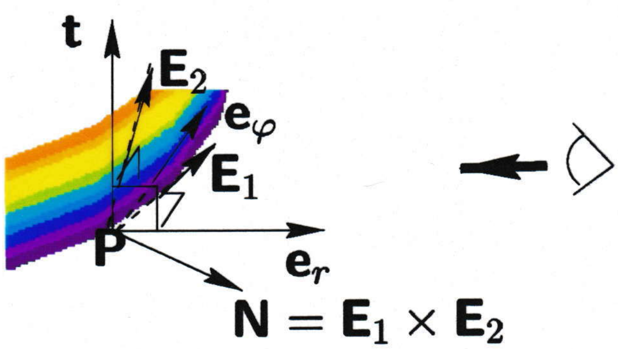
Interface

$$\mathbf{S} = \mathbf{S}(r, s, t) \qquad \varphi = \Phi(r, s, t)$$

$$\mathbf{x} = \mathbf{OP} = \mathbf{S} = \mathbf{X}(s, t) + r\mathbf{e}_r(\Phi(r, s, t), s, t)$$

$$\Psi = r\partial\Phi/\partial r \qquad \chi = \sigma T + \Phi_s \qquad \sigma(s, t) = |\mathbf{X}_s|$$





Asymptotic Expansions solution

Unknown

$$u(r,\varphi,s,t,\epsilon)\ ? \qquad v(r,\varphi,s,t,\epsilon)\ ?$$

$$\mathbf{X}(s,t,\epsilon)\ ? \qquad \Phi(r,s,t,\epsilon)\ ?$$

"Fife scaling"

$$\begin{aligned} t &= t/\epsilon^{1/3} \\ x &= x/\epsilon^{2/3} \\ \epsilon &= \epsilon^{1/3} \\ \delta &= \delta^{1/3} \end{aligned}$$

New form of the system

$$\begin{aligned} \epsilon^2 \partial u/\partial t &= \epsilon^2 \Delta^2 u + f(u,v) \\ \partial v/\partial t &= \delta^3 \Delta^2 v + \epsilon g(u,v) \end{aligned}$$

with $\epsilon \ll 1$.

Expansions solution

- Curves and Surfaces :

$$\begin{aligned} \mathbf{X}(s,t,\epsilon) &= \mathbf{X}^{(0)}(s,t) + \epsilon \mathbf{X}^{(1)}(s,t) + \ldots \\ \Phi(r,s,t,\epsilon) &= \Phi^{(0)}(r,s,t) + \epsilon \Phi^{(1)}(r,s,t) + \ldots \\ \Psi(r,s,t,\epsilon) &= \Psi^{(0)}(r,s,t) + \epsilon \Psi^{(1)}(r,s,t) + \ldots \end{aligned}$$

- Outer region:

$$
\begin{aligned}
u(r,s,\varphi,t,\epsilon) &= u^{(0)}(r,s,\varphi,t) + \epsilon u^{(1)}(r,s,\varphi,t) + \ldots \\
v(r,s,\varphi,t,\epsilon) &= v^{(0)}(r,s,\varphi,t) + \epsilon v^{(1)}(r,s,\varphi,t) + \ldots
\end{aligned}
$$

- Interface boundary layer :

$(\tilde{r}, s, \xi)$ coordinates

$$
\mathbf{x} = \mathbf{OM} = \mathbf{X}(s,t) + \tilde{r}\mathbf{e}_r(\Phi(\tilde{r},s,t),s,t) + \xi\mathbf{N}
$$

$$
\begin{aligned}
u(\tilde{r},s,\bar{\xi},t,\epsilon) &= u^{i(0)}(\tilde{r},s,\bar{\xi},t) + \epsilon u^{i(1)}(\tilde{r},s,\bar{\xi},t) + \ldots \\
v(\tilde{r},s,\bar{\xi},t,\epsilon) &= v^{i(0)}(\tilde{r},s,\bar{\xi},t) + \epsilon v^{i(1)}(\tilde{r},s,\bar{\xi},t) + \ldots
\end{aligned}
$$

where $\bar{\xi} = \xi/\epsilon$.

- Core boundary layer :

$$
\begin{aligned}
u(r,s,\bar{r},t,\epsilon) &= u^{c(0)}(r,s,\bar{r},t) + \epsilon u^{c(1)}(r,s,\bar{r},t) + \ldots \\
v(r,s,\bar{r},t,\epsilon) &= v^{c(0)}(r,s,\bar{r},t) + \epsilon v^{c(1)}(r,s,\bar{r},t) + \ldots
\end{aligned}
$$

where $\bar{r} = r/\epsilon$.

Equations in local coordinates

Coordinates (r, φ, s)

$$\epsilon^2 \partial u/\partial t = \epsilon^2 \left(\frac{1}{rh_3}\frac{\partial}{\partial r}(rh_3\frac{\partial u}{\partial r}) + \frac{1}{r^2 h_3}\frac{\partial}{\partial \varphi}(h_3\frac{\partial u}{\partial \varphi}) + \mathcal{H}(\mathcal{H}u) \right) + f(u,v)$$

$$+\epsilon^2 \left(\dot{\mathbf{X}} \cdot \mathbf{e}_r \frac{\partial u}{\partial r} + (\dot{\mathbf{X}} + r\dot{\mathbf{e}}_r) \cdot \mathbf{e}_\varphi \frac{1}{r}\frac{\partial u}{\partial \varphi} + (\dot{\mathbf{X}} + r\dot{\mathbf{e}}_r) \cdot \mathbf{t}\mathcal{H}u \right)$$

$$\partial v/\partial t = \delta^3 \left(\frac{1}{rh_3}\frac{\partial}{\partial r}(rh_3\frac{\partial v}{\partial r}) + \frac{1}{r^2 h_3}\frac{\partial}{\partial \varphi}(h_3\frac{\partial v}{\partial \varphi}) + \mathcal{H}(\mathcal{H}v) \right) + \epsilon g(u,v)$$

$$+\dot{\mathbf{X}} \cdot \mathbf{e}_r \frac{\partial v}{\partial r} + (\dot{\mathbf{X}} + r\dot{\mathbf{e}}_r) \cdot \mathbf{e}_\varphi \frac{1}{r}\frac{\partial v}{\partial \varphi} + (\dot{\mathbf{X}} + r\dot{\mathbf{e}}_r) \cdot \mathbf{t}\mathcal{H}v$$

$$\mathcal{H}u = \left[\left(\frac{\partial u}{\partial s} \right)_\varphi - \sigma T \frac{\partial u}{\partial \varphi} \right] / h_3$$

Coordinates $(\tilde{r}, \xi, s)$

$$
\begin{aligned}
\epsilon^2 \partial u / \partial t &= \epsilon^2 \left(\Delta^s u + \frac{1}{g^{1/2}} \frac{\partial}{\partial \xi} (g^{1/2} \frac{\partial u}{\partial \xi}) \right) + f(u, v) \\
&\quad + \epsilon^2 \left(\dot{\mathbf{X}} + \tilde{r} \left(\dot{\Phi} \mathbf{e}_\varphi + \dot{\mathbf{e}_r} \right) + \xi \dot{\mathbf{N}} \right) \cdot \left(\nabla^s u + \mathbf{N} \frac{\partial u}{\partial \xi} \right) \\
\partial v / \partial t &= \delta^3 \left(\Delta^s v + \frac{1}{g^{1/2}} \frac{\partial}{\partial \xi} (g^{1/2} \frac{\partial v}{\partial \xi}) \right) + \epsilon g(u, v) \\
&\quad + \left(\dot{\mathbf{X}} + \tilde{r} \left(\dot{\Phi} \mathbf{e}_\varphi + \dot{\mathbf{e}_r} \right) + \xi \dot{\mathbf{N}} \right) \cdot \left(\nabla^s v + \mathbf{N} \frac{\partial v}{\partial \xi} \right) \\
\nabla^s u &= \mathbf{E}_1^s \left(\frac{H_{11}}{g} \frac{\partial u}{\partial \tilde{r}} + \frac{H_{12}}{g} \frac{\partial u}{\partial s} \right) + \mathbf{E}_2^s \left(\frac{H_{21}}{g} \frac{\partial u}{\partial \tilde{r}} + \frac{H_{22}}{g} \frac{\partial u}{\partial s} \right) \\
\Delta^s u &= \frac{1}{g^{1/2}} \left(\frac{\partial}{\partial \tilde{r}} (g^{1/2} g^{11} \frac{\partial u}{\partial \tilde{r}}) + \frac{\partial}{\partial s} (g^{1/2} g^{22} \frac{\partial u}{\partial s}) \right) \\
&\quad \frac{1}{g^{1/2}} \left(+ \frac{\partial}{\partial \tilde{r}} (g^{1/2} g^{12} \frac{\partial u}{\partial s}) + \frac{\partial}{\partial s} (g^{1/2} g^{12} \frac{\partial u}{\partial \tilde{r}}) \right)
\end{aligned}
$$

Solution

Outer

- Leading order :

$$\begin{aligned} f(u^{(0)}, v^{(0)}) &= 0 \\ \partial v^{(0)}/\partial t &= 0 \end{aligned}$$

- First order :

$$\begin{aligned} \partial v^{(1)}/\partial t &= g(h^{\pm}(v^{(0)}), v^{(0)}) = g^{\pm}(v^{(0)}) \\ u^{(1)} &= -v^{(1)} \frac{f_v(u^{(0)}, v^{(0)})}{f_u(u^{(0)}, v^{(0)})} \end{aligned}$$

Core

- Leading order :

$$\dot{\mathbf{X}}^{(0)} = 0$$

Interface

- Leading order :

$$\frac{\partial v^{i(0)}}{\partial \bar{\xi}} = 0$$

$$\frac{\partial^2 u^{i(0)}}{\partial \bar{\xi}^2} + f(u^{i(0)}, v^{i(0)}) = 0$$

The matching law leads us to

$$
\begin{aligned}
v^{i(0)} &= v^{(0)}(\xi = 0) \\
u^{i(0)}(\bar{\xi} = \pm\infty) &= u^{\pm(0)}(\xi = 0)
\end{aligned}
$$

$$
v^{i(0)} = v^s = a/2 - b \qquad v^{(0)} = v^s
$$

$$
u^{i(0)} = \left(1 + e^{-\bar{\xi}/\sqrt{2}}\right)^{-1}
$$

- First order :

$$
\begin{aligned}
\frac{\partial^2 u^{i(1)}}{\partial \bar{\xi}^2} + u^{i(1)} f_u(u^{i(0)}, v^s) &= -v^{i(1)} f_v(u^{i(0)}, v^s) \\
&\quad -r\dot{\Phi}\mathbf{e}_\varphi \cdot \mathbf{N}\frac{\partial u^{i(0)}}{\partial \bar{\xi}} \\
&\quad +2H\frac{\partial u^{i(0)}}{\partial \bar{\xi}}
\end{aligned}
$$

Solution and matching ($1/u$ and $1/(1-u)$) :

$$
\partial v^{(1)}/\partial t = g^{\pm}(v^{(0)})
$$

$$
-\frac{h_3^s r}{\sqrt{g^s}}\dot{\Phi} = -\frac{1}{\sqrt{2}}v^{i(1)} + 2H
$$

$$
\begin{aligned}
h_3^s &= \sigma(1 - rK\cos(\Phi)) \\
g^s &= (1 + \psi^2) h_3^{s\,2} + r^2\chi^2
\end{aligned}
$$

$$
\begin{aligned}
H = & \frac{1}{2g^{s\,3/2}}[a_1(1+\Psi^2) - 2a_2 r\chi\Psi] \\
& -(h_3^{s\,2} + r^2\chi^2)\frac{h_3^s}{r}\frac{(\Psi^3 + \Psi + r\Psi_r)}{2g^{s\,3/2}}
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= r\chi^2[-h_3^{s\,2}\Psi + r\sigma K\sin\Phi] \\
&\quad + h_3^{s\,2}\sigma K[\Psi\cos\Phi + \sin\Phi] \\
&\quad + r\chi\frac{\partial h_3^s}{\partial s} - r h_3^s \chi_s \\
a_2 &= -\chi[h_3^s\Psi^2 + \sigma - \sigma K r\Psi\sin\Phi] - h_3^s\Psi_s
\end{aligned}
$$

Coordinates $(r, s, \tilde{\varphi})$ that fixes the interface

$$
\varphi = \tilde{\varphi} + \Phi^+(r,s,t) + (\Phi^-(r,s,t) - \Phi^+(r,s,t) - \pi)\sin^2\frac{\tilde{\varphi}}{2}
$$

$$
\tilde{\varphi} = 0 \qquad \tilde{\varphi} = \pi
$$

Then :

$$
\frac{\partial v^{(1)}}{\partial t} = g^{\pm} + \frac{\dot{\varphi}}{h}\frac{\partial v^{(1)}}{\partial \tilde{\varphi}}
$$

Stationnary solutions

$$\omega(\epsilon) \quad = \quad \omega^{(0)} + \epsilon\omega^{(1)} + \ldots$$

Simplifications

$$K = T = 0 \qquad h_3^s = \sigma = 1 \qquad \chi = \tau$$

Outer solution

$$-\omega^{(0)}\partial v^{(1)}/\partial\varphi = g^{\pm}(v^{(0)})$$

$$\begin{aligned}(1 + r^2\tau^2)\partial\Psi/\partial r \quad = \quad & -\frac{\Psi(1+\Psi^2)(1+r^2\tau^2)}{r} \\ & +r(1 + r^2\tau^2 + \Psi^2) \\ & -B(1 + r^2\tau^2 + \Psi^2)^{3/2} \\ & +\Psi(1+\Psi^2)r\tau^2\end{aligned}$$

where the lengths r is the previous multiplied by $\sqrt{\omega}$ and

$$B = \frac{\sqrt{2}}{2a}\frac{v^s}{\omega^{3/2}}(1 - v^s)2\pi$$

The boundary limit at infinity gives the numerical value of B.

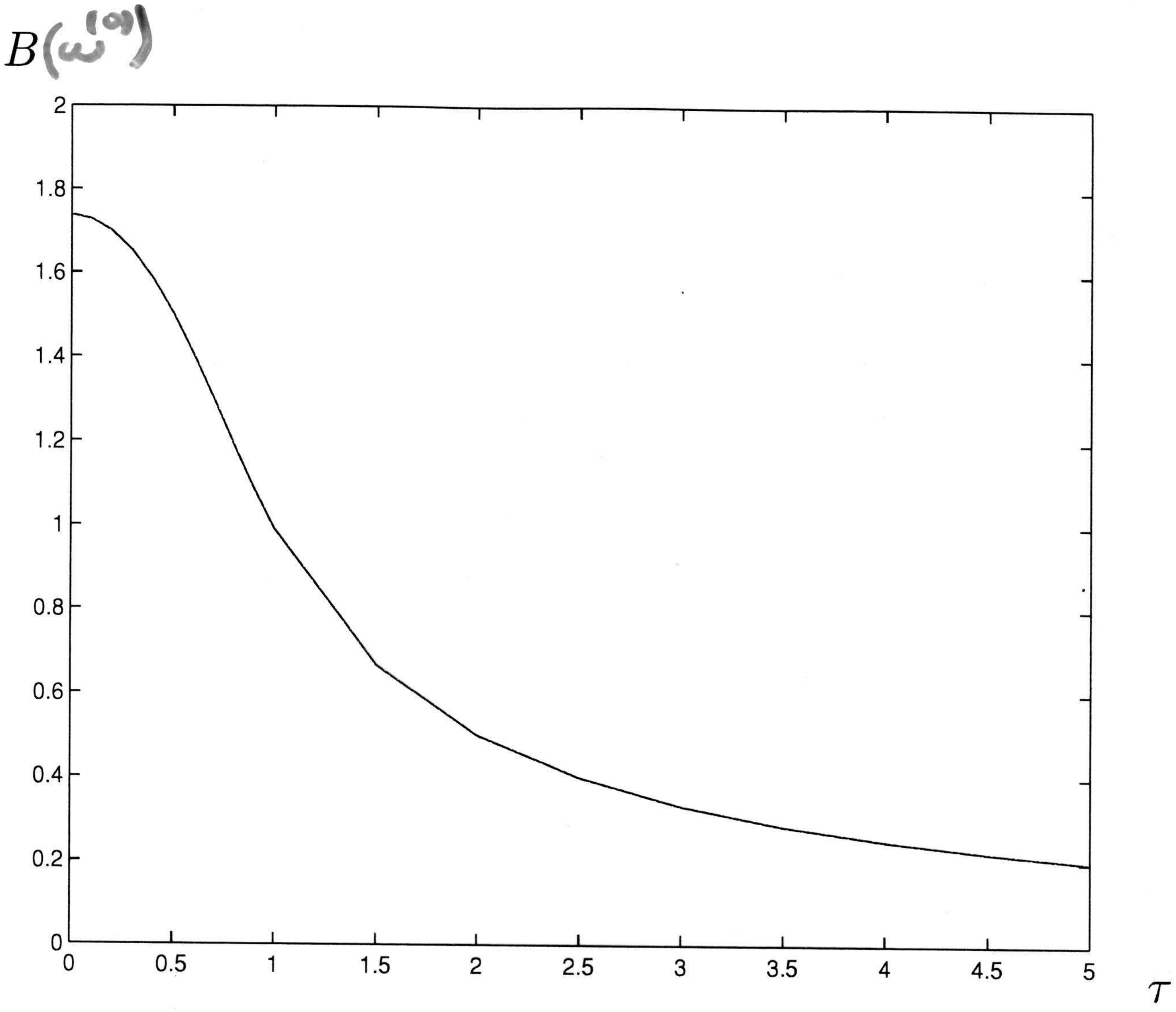


Figure 3: Evolution of B as a function of the twist τ

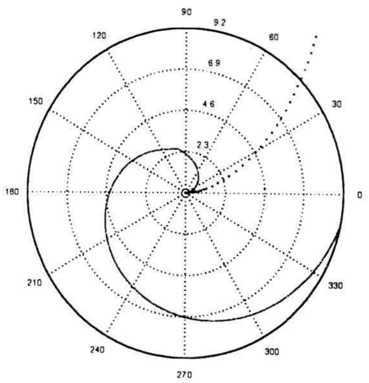
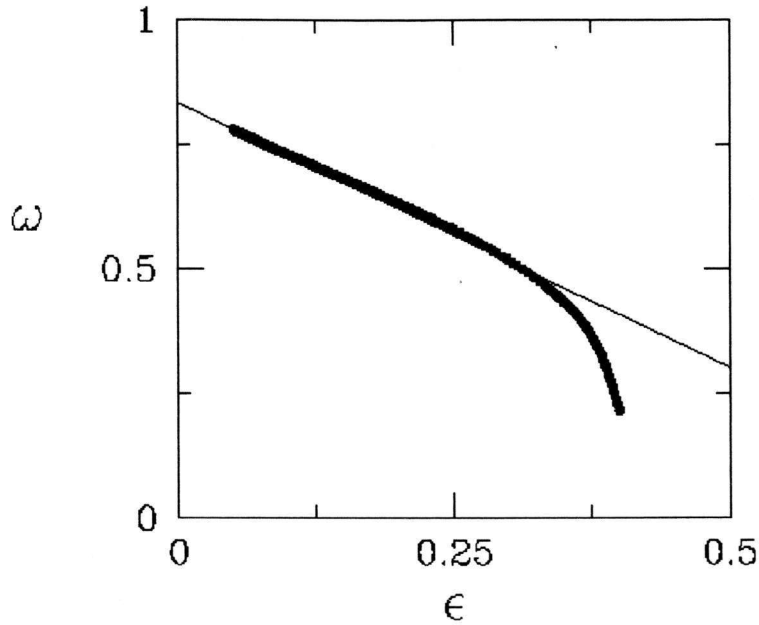


Figure 4: Shape of the spiral for $\tau = 0$ (solid line) and $\tau = 2$ (dashed)

Comparison with Direct Numerical simulations of the stationnary solution



$\tau = 0$

Figure 5: ω as regard of ϵ in the Fife scaling

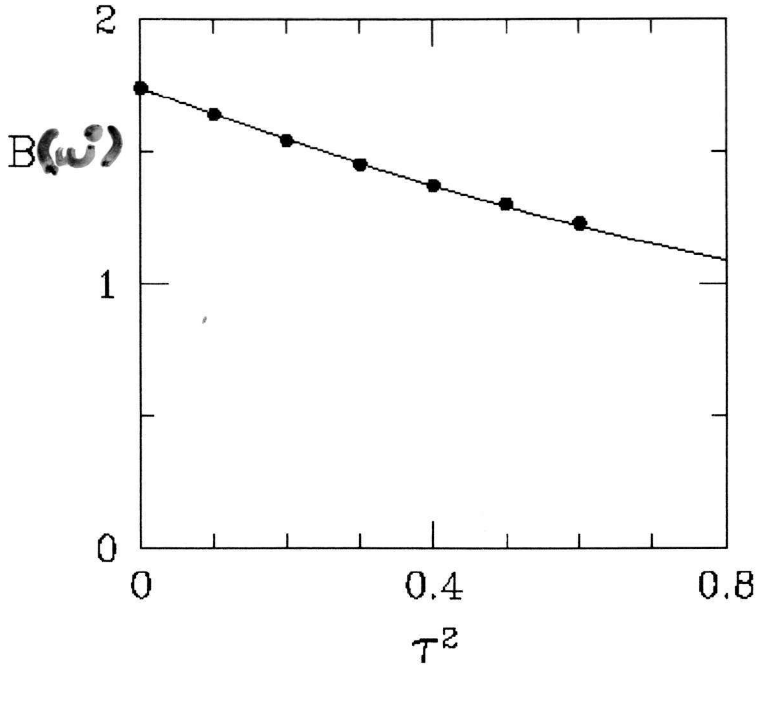


Figure 6: ω as regard of τ^2

Perspectives

- Equations for $\Psi^{(1)}(r)$, correction of the shape of the spiral, value of $\omega^{(1)}$ in 2-d stationnary

- Equations for $\Psi^{(1)}(r,t)$ in 2-d no stationnary

- Equations for $\Phi^{(1)}(r,s,t)$ in 3-d (no stationnary)

- Notice : same operator than linear stability of the leading order stationnary solution. Papers about drift : motion = matching with the core for the perturbation.

- Equation of motion for **X** as a result of a matching with the core at first order

- Particular case : $K \ll 1$

- Numerical solution of the leading order on fixed boundary

- Helix solution ?