Introduction

• Domain of study :

Incompressive laminar flows with vorticity and Re>>1

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \text{ grad } \mathbf{v} = -\text{grad } \frac{p}{\rho} + \nu \Delta \mathbf{v}$$

$$\text{div } \mathbf{v} = 0$$

• The field of Vorticity

$$\vec{\omega} = \text{rot}(\mathbf{v})$$

$$\frac{\partial \vec{\omega}}{\partial t} + \mathbf{v} \text{ grad } \vec{\omega} = \vec{\omega} \text{ grad } \mathbf{v} + \nu \Delta \vec{\omega}$$

$$\mathbf{v}(\mathbf{x}) = \text{grad}\big(\varphi(\mathbf{x})\big) \quad + \quad \frac{1}{4\pi} \iiint \frac{\vec{\omega}' \times (\mathbf{x} - \mathbf{x}')}{\left|\mathbf{x} - \mathbf{x}'\right|^3} dx'$$

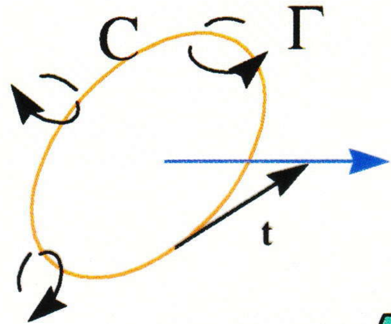
Background velocity — Induced velocity

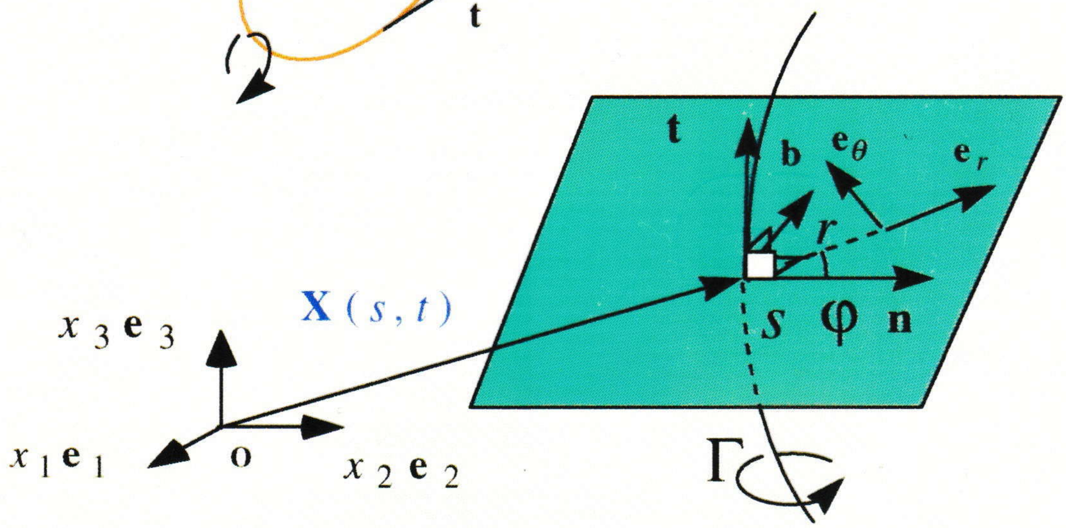
$$\Delta \varphi = 0$$

1 Equation of motion of a vortex filament

- The non thickness curved filament

$$\vec{\omega} = \Gamma \delta_C \, \mathbf{t}$$





$$\mathbf{v}(\mathbf{x}) = \frac{\Gamma}{4\pi} \int_C \frac{\mathbf{t}(s') \times (\mathbf{x} - \mathbf{X}(s'))}{\left| \mathbf{x} - \mathbf{X}(s') \right|^3} ds'$$

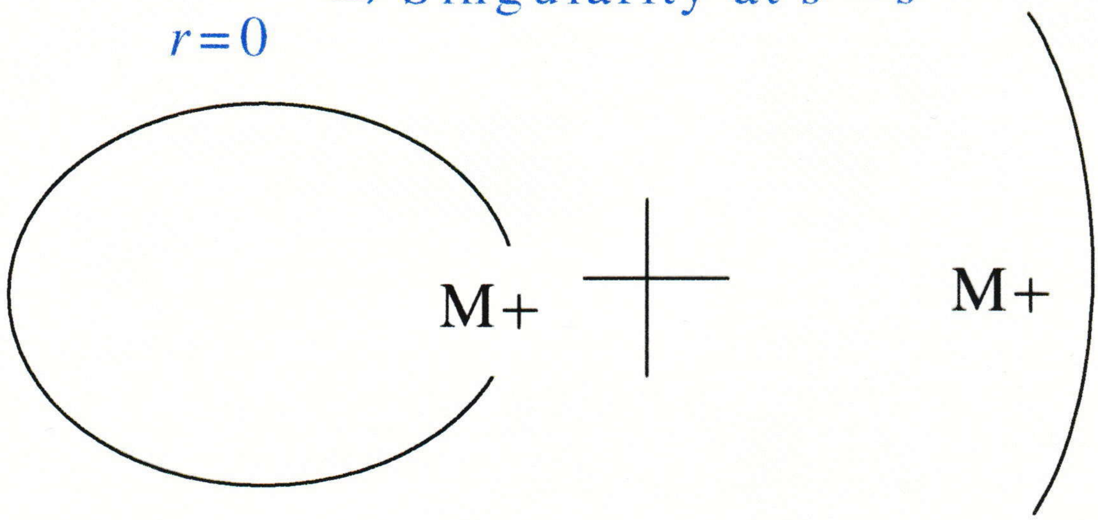
$$R_e = \frac{\Gamma}{\nu} >> 1$$

• Limit of the velocity field on the non thickness vortex filament

$$\mathbf{v}(\mathbf{x}) = \frac{\Gamma}{4\pi}\int_C \frac{\mathbf{t}(s') \times (\mathbf{x} - \mathbf{X}(s'))}{\left|\mathbf{x} - \mathbf{X}(s')\right|^3} ds'$$

$\mathbf{x} = \mathbf{X}(s)$ $\Rightarrow$ Singularity at $s' = s$

$r = 0$



$r \to 0$ with s'-s fixed $\qquad$ $r \to 0$ with $(s'$-$s)\,/\,r$ fixed

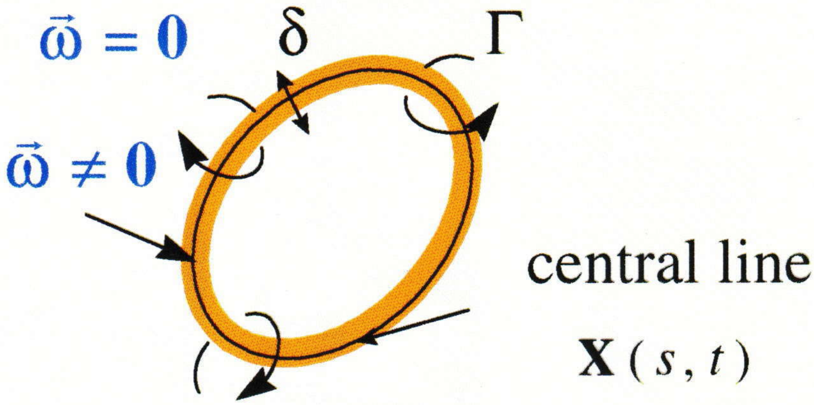
$$\mathbf{v}(r \to 0) = \frac{\Gamma}{2\pi\, r}\mathbf{e}_\theta + \frac{\Gamma K}{4\pi}\left[\ln\frac{S}{r} - 1\right]\mathbf{b} + \frac{\Gamma K}{4\pi}\cos(\varphi)\mathbf{e}_\theta + \mathbf{A} + O(r \ln r)$$

$$\text{with } \mathbf{A} = \frac{1}{4\pi}\int_C \left[\mathbf{t}(s') \times \frac{(\mathbf{X}(s) - \mathbf{X}(s'))}{\left|\mathbf{X}(s) - \mathbf{X}(s')\right|^3} - \frac{K(s)\mathbf{b}(s)}{\left|s' - s\right|} \right] ds'$$

$$\mathbf{v}(r) = \frac{\Gamma}{2\pi\, r}\mathbf{e}_\theta + \frac{\Gamma}{2\pi} K \ln\frac{1}{r}\mathbf{b} \text{ en } r \approx 0 \text{ !!!}$$

$\Rightarrow$ A thickness δ is needed

• The scales



Two lenghts :

$$\delta = \mathrm{O}(l)$$

$$R, S = \mathrm{O}(L) \qquad \text{with } \frac{l}{L} \equiv \varepsilon << 1$$

$$\mathrm{Re} = \frac{\Gamma}{\nu} = \frac{1}{\alpha^2 \varepsilon^2} >> 1 \qquad \alpha = O(1) \text{ or } \alpha = 0$$

$$\dot{\mathbf{X}} = ?$$

- Frenet formula :

$$\sigma(s,t) = |\mathbf{X}_s| \quad \mathbf{X}_s = \sigma\mathbf{t} \quad \mathbf{t}_s = \sigma K\mathbf{n}$$

$$\mathbf{n}_s = \sigma(T\mathbf{b} - K\mathbf{t}) \qquad \mathbf{b}_s = -\sigma T\mathbf{n}$$

-Dimensionless :

$$t^* = t/(L^2/\Gamma) \quad \mathbf{v}^* = \mathbf{v}/(\Gamma/L) \quad \vec{\omega}^* = \vec{\omega}/(\Gamma/L^2)$$

$$r^* = r/L \quad \mathbf{X}^* = \mathbf{X}/L \quad \sigma^* = \sigma/L \quad K^* = LK$$

$$T^* = LT \quad S^* = S/L \quad \delta^* = \delta/L$$

- Central line : $\mathbf{X} = \mathbf{X}^{(0)}(t,s) + \varepsilon\ \mathbf{X}^{(1)}(t,s) + \ldots$

$$\mathbf{v} = \dot{\mathbf{X}} + \mathbf{V}$$

• Outer and Inner limits

- Outer expansion :

• $\varepsilon \to 0$ with r fixed : *outer limit*

$$\mathbf{v}^{\text{out}} = \mathbf{v}^{\text{out}(0)}(t,r,s) + \varepsilon\ \mathbf{v}^{\text{out}(1)}(t,r,\varphi,s) + \ldots$$

- Inner expansions :

• $\varepsilon \to 0$ with $\bar{r} = r/\varepsilon$ fixed : *inner limit*

$$\mathbf{V}^{\text{inn}} = \varepsilon^{-1}\mathbf{V}^{\text{inn}(0)}(t,\bar{r},s) + \mathbf{V}^{\text{inn}(1)}(t,\bar{r},\varphi,s) + \ldots$$

$$\dot{\mathbf{X}} = f(\mathbf{X}, \mathbf{V}^{\text{inn}}) \quad ? \qquad \mathbf{V}^{\text{inn}} = ?$$

- **Equations :**
  - Outer :

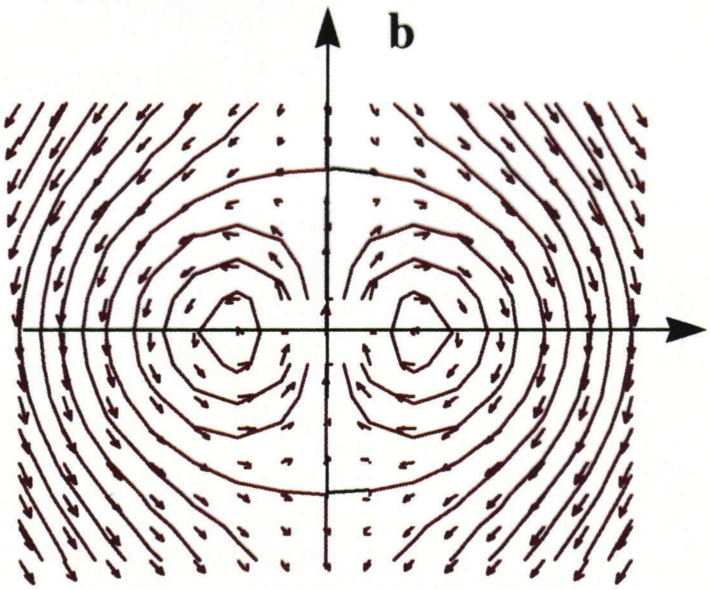
Singular line Biot & Savart integral

$$\mathbf{v}(r) = \frac{\Gamma}{2\pi r}\mathbf{e}_\theta + \frac{\Gamma}{2\pi} K \ln\frac{1}{r}\mathbf{b} \text{ en } r \approx 0 \text{ !!!}$$

$\Rightarrow$ Boundary Conditions at $\bar{r} = \infty$

  - Inner : Navier Stokes equations
    Boundary Conditions at $\bar{r} = 0$

- $\sin(\varphi)$ and $\cos(\varphi)$ part at order i=1 :



No need of Boundary Conditions at infinity !

Limit $\bar{r} \to \infty$ (Singu. Integral)

Identification
with Boundary Conditions at infinity

$\Rightarrow$

Callegari & Ting

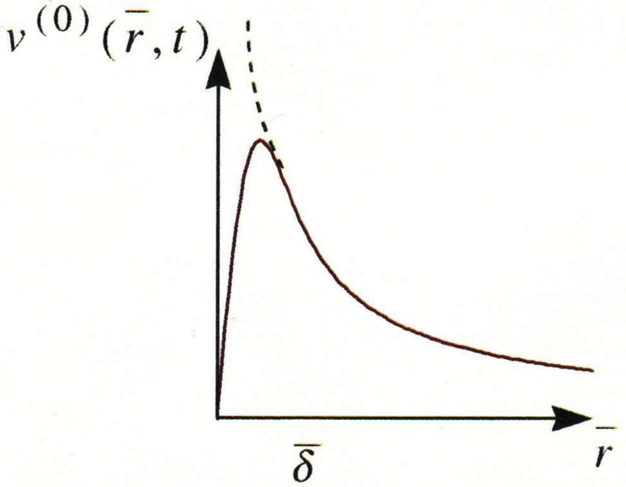
$$\dot{\mathbf{X}}^{(0)}(s,t) = \mathbf{Q} + \frac{K^{(0)}(s,t)}{4\pi}\left[-\ln\varepsilon + \ln S - 1 + C_v(t) + C_w(t)\right]\mathbf{b}^{(0)}$$

$$\mathbf{Q} = \mathbf{A} - (\mathbf{A}\cdot\mathbf{t})\,\mathbf{t}$$

$$\mathbf{A} = \frac{1}{4\pi}\int_{-\pi}^{+\pi}\sigma(s+s')\left[\frac{\mathbf{t}(s+s')\times\left(\mathbf{X}(s)-\mathbf{X}(s+s')\right)}{\left|\mathbf{X}(s)-\mathbf{X}(s+s')\right|^{3}} - \frac{K(s)\mathbf{b}(s)}{2\left|\lambda(s,s',t)\right|}\right]ds'$$

$$\lambda(s,s',t) = \int_{0}^{s+s'}\sigma(s^*,t)\,ds^*$$

- **Similar Ring**



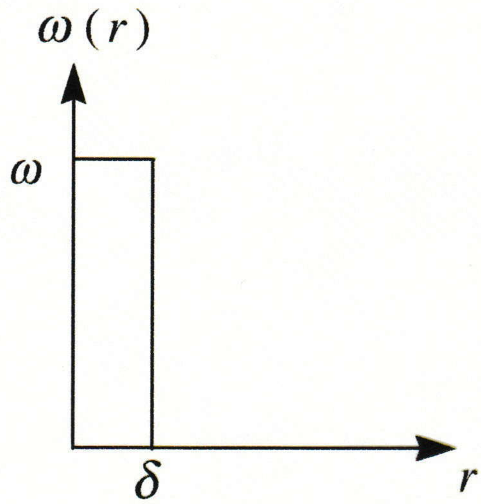
$$\bar{\delta}(t) = \left(\frac{S_0}{S}\right)^{1/2}\left(1 + 4\alpha^2\int_0^t \frac{S(t')}{S_0}\,dt'\right)^{1/2}$$

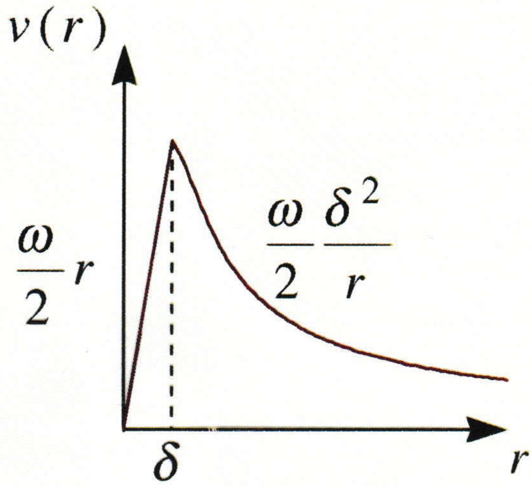
$$C_v(t) = \left[1 + \gamma - \ln 2\right]/2 - \ln\bar{\delta}$$

$S(t)$: length of the ring

- # Vorticity & Velocity fields

  - inviscid : Rankine





  - viscous: Burger