Task 2.2.3

Numerical Flowfield Computation by Simplified Methods

Adaptation of 3D Vortex Filament Methods

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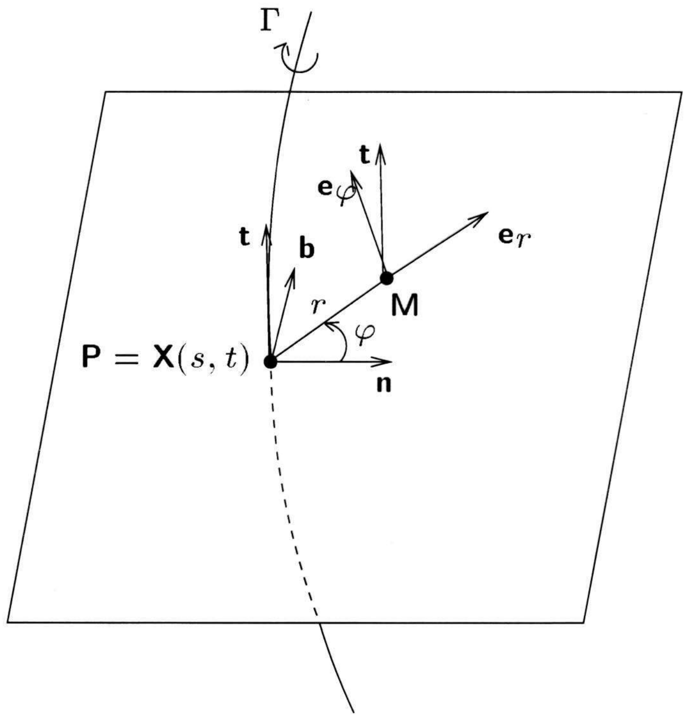
Equation of motion of slender filaments



The centerline and the local co-ordinates of the vortex filament.

Slenderness ⟹ [redacted] in the Navier Stokes Equations
⟹ Separated Slender Filaments (No reconnection)
⟹ No short-wavelength

⟹ Callegari and Ting: Siam Applied Math. 78
⟹ Klein and Knio : J. Fluid Mech. 95
⟹ Klein and Knio : J. Comput. 2000

1) The equation of Callegari and Ting

$$\partial\mathbf{X}/\partial t = \mathbf{A} + \frac{\Gamma K(s,t)}{4\pi} B\mathbf{b}(s,t),$$

where

$$\begin{aligned}
\mathbf{A}(s,t) &= \frac{\Gamma}{4\pi}\int_{-\pi}^{+\pi} \sigma(s+s',t)\mathbf{N}\mathrm{d}s', \\
\mathbf{N} &= \frac{\mathbf{t}(s+s',t)\times(\mathbf{X}(s,t)-\mathbf{X}(s+s',t))}{|\mathbf{X}(s,t)-\mathbf{X}(s+s',t)|^3} \\
&\quad -\frac{K(s,t)\mathbf{b}(s,t)}{2|\lambda(s,s',t)|}, \\
\sigma(s,t) &= |\partial\mathbf{X}/\partial s|, \\
\lambda(s,s',t) &= \int_{s}^{s+s'} \sigma(s^*,t)\mathrm{d}s^*,
\end{aligned}$$

and

$$B = -\log\epsilon + \log(S) - 1 + C_v(t) + C_w(t)$$

K: local curvature

S: length of the filament

ϵ: small dimensionless thickness

Similar vortex core

$$C_v(t) = [1+\gamma-\ln 2]/2 - \ln(\bar\delta),$$
$$C_w(t) = -2(S_0/S)^4(m_0/(\Gamma\bar\delta))^2,$$

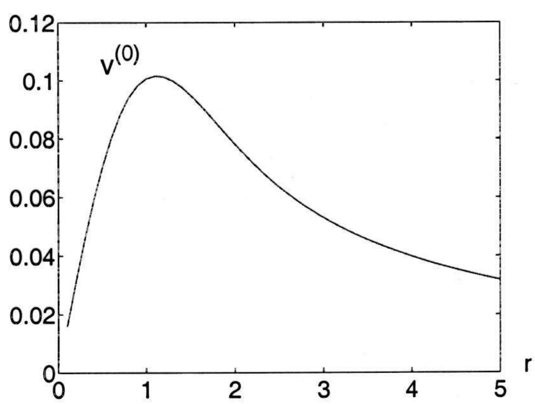
where

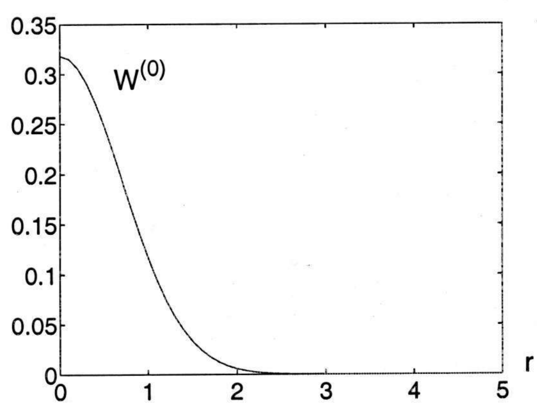
$$\bar\delta^2(t) = \bar\delta_0^2\left(\frac{S_0}{S(t)}\right)\left(1+\frac{\bar\delta_{\bar\nu}^2}{\bar\delta_0^2}\right)$$
$$\bar\delta_{\bar\nu}^2 = 4\bar\nu\int_0^t \frac{S(t')}{S_0}\mathrm{d}t',$$

γ= Euler number. Subscript 0 stands for initial.
$\bar\delta = \delta/\epsilon$: the stretched radius
$\bar\nu = \nu/\epsilon^2$ is the stretched kinematic viscosity.
m_0: the initial axial flux of the ring.

Circumferential $v^{(0)}$ and axial $w^{(0)}$ velocities:

$$v^{(0)} = \frac{\Gamma}{2\pi\bar r}\left[1-e^{-(\bar r/\bar\delta)^2}\right], \qquad w^{(0)} = \frac{m_0}{\pi\bar\delta^2}\left(\frac{S_0}{S}\right)^2 e^{-(\bar r/\bar\delta)^2},$$





$\bar r = r/\epsilon$: the stretched radial distance to the filament,

2) M1 method of Knio and Klein

$$\partial \mathbf{X}/\partial t = \mathbf{v}_{\sigma_1} + (\mathbf{v}_{\sigma_1} - \mathbf{v}_{\sigma_2}) \frac{\log(\sigma_1/\delta^{ttm})}{\log(\sigma_2/\sigma_1)}$$

where

$$\begin{aligned}
\mathbf{v}_x &= \frac{\Gamma}{4\pi} \int_C \sigma(s', t) \mathbf{N}_x \mathrm{d}s', \\
\mathbf{N}_x &= \frac{\mathbf{t}(s', t) \times (\mathbf{X}(s, t) - \mathbf{X}(s', t))}{\left[|\mathbf{X}(s, t) - \mathbf{X}(s', t)|^2 \right]^{3/2}} \kappa\left(\frac{|\mathbf{X}(s, t) - \mathbf{X}(s', t)|}{x}\right) \\
\kappa(r) &= \tanh(r^3)
\end{aligned}$$

and

$$\begin{aligned}
\sigma_1 &= 3\sigma_{max}, \\
\sigma_2 &= 2\sigma_{max}, \\
\sigma_{max} &= \mathrm{d}s \max_{s \in [0, 2\pi]} \sigma(s, t). \\
\delta^{ttm} &= \epsilon \exp\left(C^{ttm} + 1 - C_v(t) - C_w(t) \right),
\end{aligned}$$

With the choice of $\kappa(r) = \tanh(r^3)$, the C^{ttm} constant is $C^{ttm} = -0.4202$.

EZ-vortex[1]: a Slender Vortex Filament solver (SVF)

1) Input: Initial position of the filaments

2) Solver: SVF solver for *closed* or *open* filaments

- Equation:
  - The Callegari and Ting Equation (Implicit stepping)
  - The M1 method of Knio and Klein (Adams-Bashforth or Implicit)
- Spatial derivatives: Finit. Diff. or Spectral
- Core: similar core or non-similar (Laguerre series)
- Inviscid or viscous

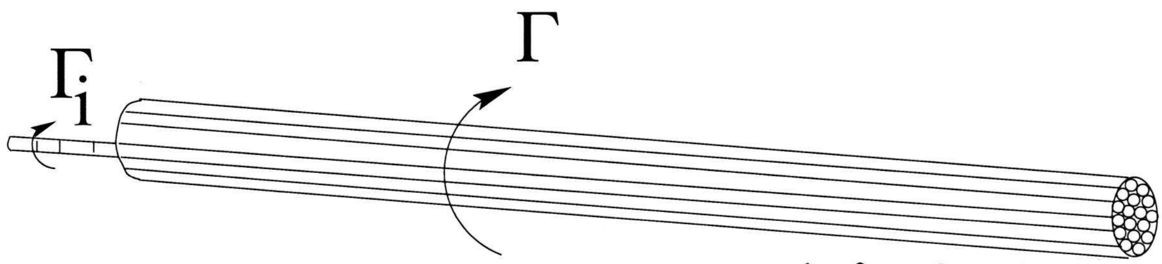
3) Output:

- Run-time drawing with OpenGL (SGI or linux)
- File *history.dat* and mode 'Visualisation' of this file
- Movie of the simulation

[1] D.Margerit, A.Giovannini, P. Brancher: EZ-vortex documentation: a Slender Vortex Filament solver

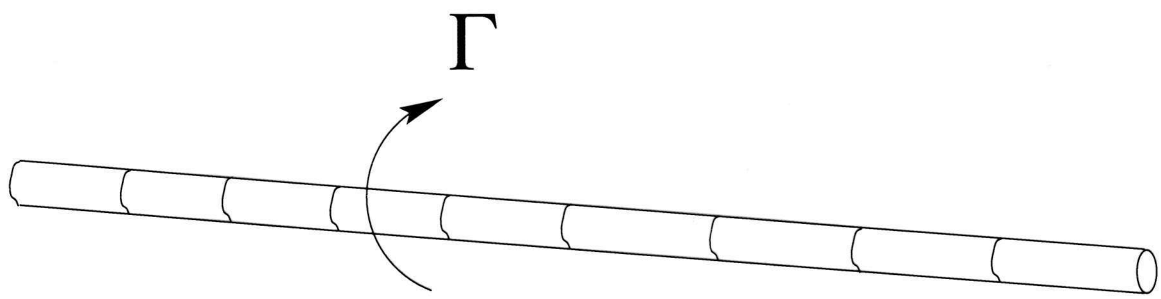
3D Numerical Vortex Methods

1) Vortex Blob or Vortex filament Methods



+ No numerical diffusion. Vorticity domai. << Flow dom.
+ Great Nbr filaments(or blobs)/section to converge
+ Stiff when the thickness is small (boundary layer)
+ Viscous diffusion computed by:
a *random walk* or a *deterministic technique*
+ N-body problem $\Longrightarrow$ Fast solvers

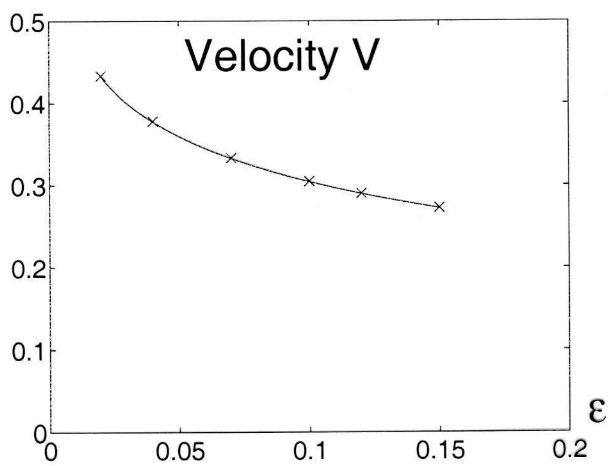
2) Slender Vortex filament solver:



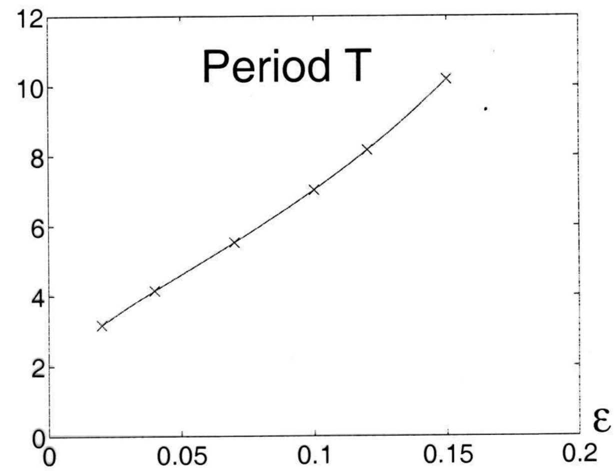
+ Nbr filaments(or blobs)/section =1
+ Not stiff with thickness:
boundary layer solved theoretically
+ Viscous diffusion solved theoreticaly
$\Longrightarrow$ Fast and accurate
but
$\Longrightarrow$ Small thickness, no reconnection, no short wavelengths

Validation

Linear stability analysis of a preturbed circular vortex ring (similar vortex and inviscid):

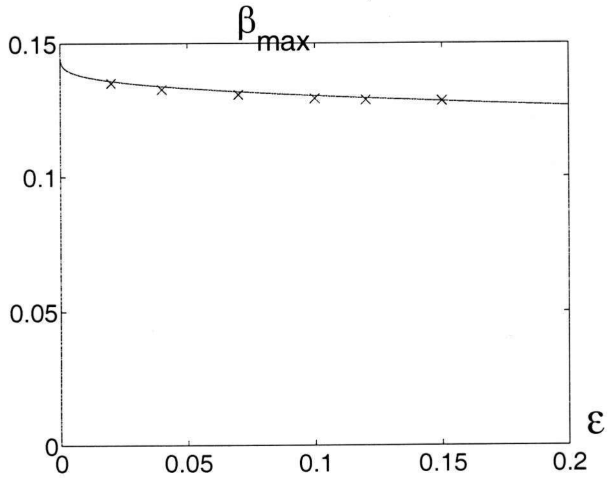


Velocity $V = f(\epsilon)$

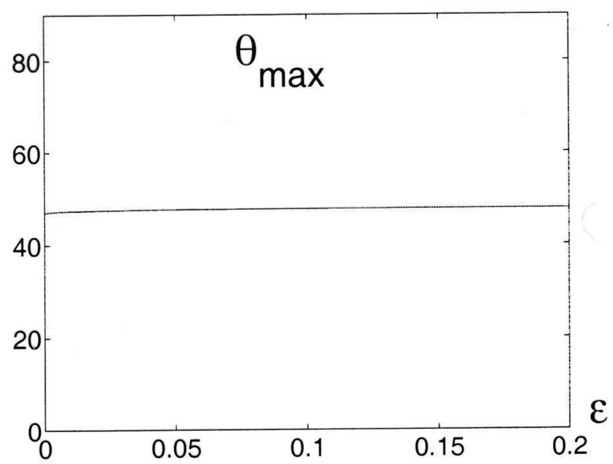


Period $T = f(\epsilon)$ for mode 3

Linear stability analysis of a pair of contra-rotating vortex filaments (similar vortex and inviscid):



Ampl. rate $\beta_{max} = f(\epsilon)$



Planar angle $\theta_{max} = f(\epsilon)$

Movies

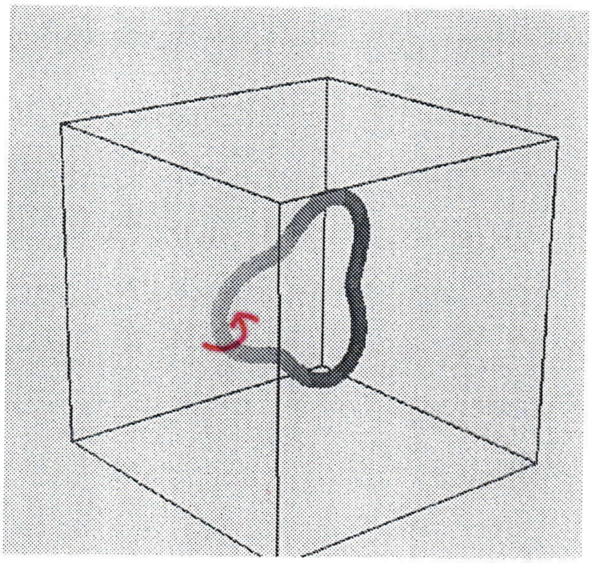


Figure 1:

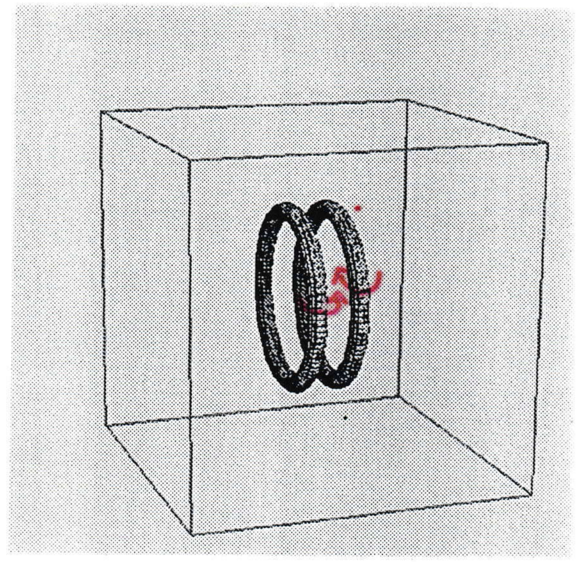


Figure 2:

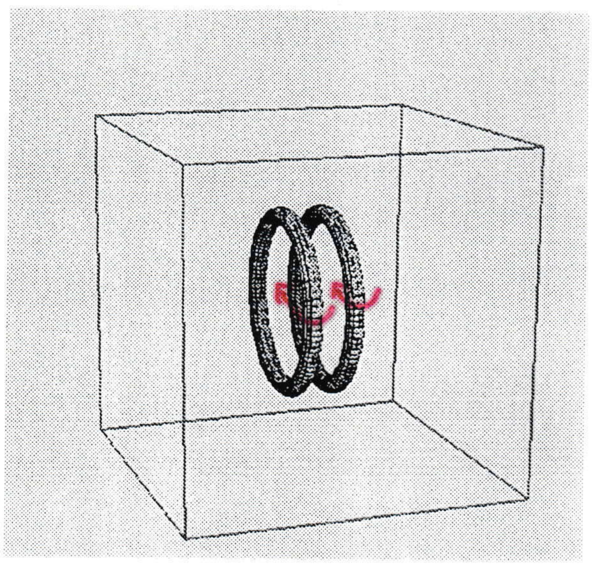
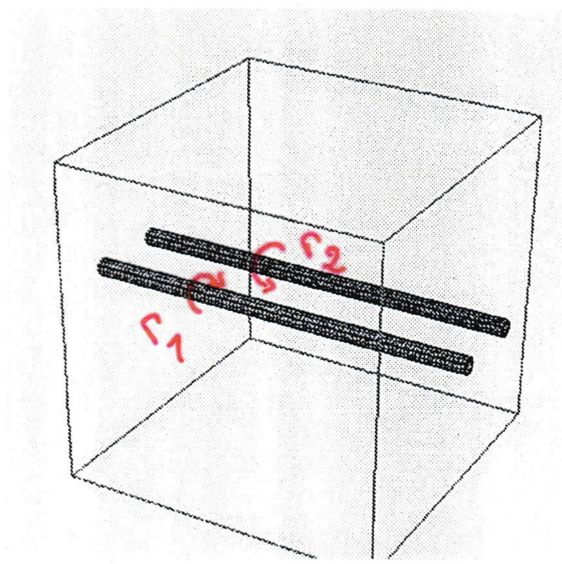


Figure 3:



http://www.maths.warwick.ac.uk/~dmargeri/movie$i$.html