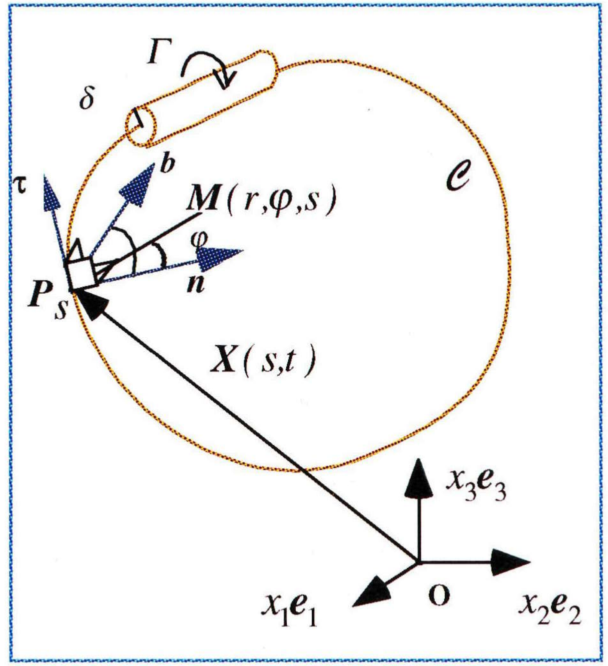
The inner velocity field

Definitions and Notations

- $\mathbf{X}(s,t)$: central line
- $\mathbf{M}(r,\varphi,s)$: local curvilinear co-ordinate
- $\mathbf{x}=\mathbf{OM}=\mathbf{X}(s,t)+r\mathbf{r}(\varphi,s,t)$: change of co-ordinates
- δ : thickness
- L : order of other length scales
- $\varepsilon=\delta(t=0)/L$: small parameter
- $\varepsilon\to 0$ with r fixed : *outer limit*
- $\varepsilon\to 0$ with $\bar{r}=r/\varepsilon$ fixed : *inner limit*
- $\mathbf{v}_{Biot}(r,s,\varphi,t,\varepsilon)$, $\boldsymbol{\omega}(\bar{r},s,\varphi,t,\varepsilon)$: velocity and vorticity fields
- Biot & Savart formula : $h_3'=\sigma(s',t)\ (1-K(s',t)\varepsilon\bar{r}'\cos(\varphi'))$ K curvature



$$\mathbf{v}_{Biot}(r,s,\varphi,t,\varepsilon)=\frac{1}{4\pi}\iiint\frac{\bar{\boldsymbol{\omega}}(\bar{r}',s',\varphi',t,\varepsilon)\wedge\left[(\mathbf{X}(s,t,\varepsilon)+r\,\mathbf{r}(s,\varphi,t,\varepsilon))-(\mathbf{X}(s',t,\varepsilon)+\varepsilon\,\bar{r}'\,\mathbf{r}')\right]}{\left|(\mathbf{X}(s,t,\varepsilon)+r\,\mathbf{r}(s,\varphi,t,\varepsilon))-(\mathbf{X}(s',t,\varepsilon)+\varepsilon\,\bar{r}'\,\mathbf{r}')\right|^3}h_3'\,\bar{r}'\,d\bar{r}'\,d\varphi'\,ds'$$

# of a slender vortex ring

Outer expansion of Biot & Savart

velocity induced by $\frac{1}{\varepsilon^2}\overline{\omega}0(\bar{r},a,\varphi)$

$$\mathbf{v}_0^{out}(r,\varphi,a,\varepsilon) = \mathbf{v}_0^{out(0)}(r,\varphi,a) + \varepsilon\, \mathbf{v}_0^{out(1)}(r,\varphi,a) + O(\varepsilon^2)$$

$$\mathbf{v}_0^{out(0)}(r,\varphi,a) = \frac{1}{4\pi}\int_{\mathcal{C}} \frac{\boldsymbol{\tau}(a') \wedge (\mathbf{x} - \mathbf{X}(a'))}{\left|\mathbf{x} - \mathbf{X}(a')\right|^3} da'$$

$$\mathbf{v}_0^{out(1)}(r,\varphi,a) = \frac{1}{4\pi}\iiint \frac{\overline{\omega}0' \wedge (\mathbf{x} - \mathbf{X}(a'))}{\left|\mathbf{x} - \mathbf{X}(a')\right|^3} \bar{r}'^2 K(a') \cos\varphi'\, d\bar{r}'\, d\varphi'\, da'$$

$$- \frac{1}{4\pi}\iiint 3\frac{\left[\overline{\omega}0' \wedge (\mathbf{x} - \mathbf{X}(a'))\right]\left(\mathbf{r}' \circ (\mathbf{x} - \mathbf{X}(a'))\right)}{\left|\mathbf{x} - \mathbf{X}(a')\right|^5} \bar{r}'^2\, d\bar{r}'\, d\varphi'\, da'$$

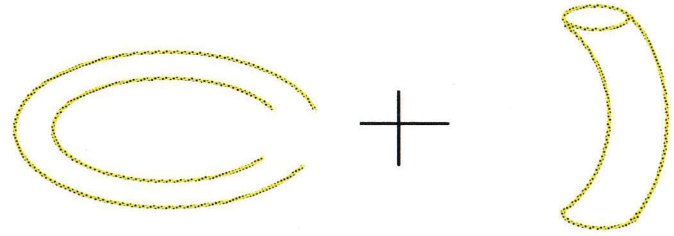
$$- \frac{1}{4\pi}\iiint \frac{\left(\mathbf{r}' \wedge \overline{\omega}0'\right)}{\left|\mathbf{x} - \mathbf{X}(a')\right|^3} \bar{r}'^2\, d\bar{r}'\, d\varphi'\, da'$$

with : $\mathbf{x} = \mathbf{X}(a) + r\mathbf{r}(\varphi,a)$

MARGERIT Daniel
LEMTA
2 avenue de la forêt de Haye BP 160
54504 Vandoeuvre les Nancy France
Daniel.Margerit@ensem.u-nancy.fr
http://www.ensem.u-nancy.fr/Ensem/laboratoires/lemta/danielf.html

Inner expansion of Biot & Savart

The schema for the expansion of the singular integral



$\varepsilon \to 0$ with a' fixed $\qquad\qquad$ $\varepsilon \to 0$ with $\frac{a'}{\varepsilon}$ fixed

$$\mathbf{v}_0^{inn}(\bar r,a,\varphi) = \frac{-1}{2\pi\varepsilon}\iint \bar{\boldsymbol\omega}_0(a,\bar r',\varphi') \wedge \frac{(-\bar r\,\mathbf{r}(\varphi,a)+\bar r'\,\mathbf{r}(\varphi',a))}{k^2}\bar r'\,d\bar r'\,d\varphi' + \frac{K_0}{4\pi}\left[\ln\frac{S_0}{\varepsilon}-1\right]\mathbf{b}+\mathbf{A}$$

$$-\frac{1}{4\pi}\iint\left[\bar{\boldsymbol\omega}_{0a}(a,\bar r',\varphi') \wedge \boldsymbol\tau + \bar{\boldsymbol\omega}_0(a,\bar r',\varphi') \wedge \frac{K_0}{2}\mathbf{n}\right]\ln\frac{1}{k^2}\bar r'\,d\bar r'\,d\varphi'$$

$$-\frac{1}{2\pi}\iint\frac{\bar{\boldsymbol\omega}_0(a,\bar r',\varphi') \wedge \left[\bar r'\,\mathbf{r}(\varphi',a)-\bar r\,\mathbf{r}(\varphi,a)\right]}{k^2}\left[-K_0\,\bar r'\cos(\varphi')\right]\bar r'\,d\bar r'\,d\varphi'$$

$$-\frac{1}{2\pi}\iint\frac{\bar{\boldsymbol\omega}_0(a,\bar r',\varphi') \wedge \left[\bar r'\,\mathbf{r}(\varphi',a)-\bar r\,\mathbf{r}(\varphi,a)\right]}{k^2}\left(\frac{K_0}{2}\right)\left[\bar r'\cos\varphi'+\bar r\cos\varphi\right]\bar r'\,d\bar r'\,d\varphi'$$

$$-\frac{1}{2\pi}\iint\frac{\bar{\boldsymbol\omega}_0(a,\bar r',\varphi') \wedge \left[T_0\bar r\,\bar r'\sin(\varphi-\varphi')\,\boldsymbol\tau\right]}{k^2}\bar r'\,d\bar r'\,d\varphi' \quad + O(\varepsilon\ln\varepsilon)_{\text{in maple}} \quad + O(\varepsilon)_{\text{in maple}}$$

with $\quad k^2 = \bar r^2 + \bar r'^2 - 2\bar r\bar r'\cos(\varphi-\varphi')$

$$\mathbf{A}(a) = \frac{1}{4\pi}\int_{-S_0/2}^{+S_0/2}\frac{\boldsymbol\tau(a+\bar a)\wedge(\mathbf{X}_0(a)-\mathbf{X}_0(a+\bar a))}{\left|\mathbf{X}_0(a)-\mathbf{X}_0(a+\bar a)\right|^3}-\frac{K_0(a)\mathbf{b}(a)}{2|\bar a|}\,d\bar a$$

Limit of inner expansion at infinity

$$\mathbf{v}_0^{inn}(\bar r\to\infty,\varphi,a) = \frac{1}{\varepsilon}\mathbf{v}_0^{inn(0)}(\bar r\to\infty,\varphi,a)+\ln\varepsilon\,\mathbf{v}_0^{inn(01)}(\bar r\to\infty,\varphi,a)+\mathbf{v}_0^{inn(1)}(\bar r\to\infty,\varphi,a)$$
$$+\varepsilon\ln\varepsilon\,\mathbf{v}_0^{inn(12)}(\bar r\to\infty,\varphi,a)+\varepsilon\mathbf{v}_0^{inn(2)}(\bar r\to\infty,\varphi,a)+O(\varepsilon^2\ln\varepsilon)$$

if $\boldsymbol{\omega}_0=\omega_2(\bar r)\boldsymbol{\theta}+\omega_3(\bar r)\boldsymbol{\tau}$ with (Maple) : $\mathbf{v}_0^{inn(0)}(\bar r\to\infty,\varphi,a)=\frac{1}{2\pi}\frac{\boldsymbol{\theta}}{\bar r}+\frac{I_1}{\bar r^2}+O(\frac{1}{\bar r^3})$

$$\mathbf{v}_0^{inn(1)}(\bar r\to\infty,\varphi,a)=\frac{K}{4\pi}\left[\ln\frac{S}{\bar r}-1\right]\mathbf{b}+\frac{K}{4\pi}\cos\varphi\boldsymbol{\theta}+A+\frac{I_2}{\bar r}+O(\frac{1}{\bar r^2})\qquad \mathbf{v}_0^{inn(01)}(\bar r\to\infty,\varphi,a)=-\frac{K}{4\pi}\mathbf{b}\quad ,\dots$$

We found that : $\mathbf{v}_0^{inn}(\bar r\to\infty,\varphi,a) = \mathbf{v}_0^{out}(r\to 0,\varphi,a)$ and noticed that :

$$\mathbf{A}(a)=\lim_{s\to 0}\left(\frac{1}{4\pi}\int_{[-S_0/2,s]\cup[s,+S_0/2]}\frac{\boldsymbol{\tau}(a+\bar a)\wedge(\mathbf{X}_0(a)-\mathbf{X}_0(a+\bar a))}{\left|\mathbf{X}_0(a)-\mathbf{X}_0(a+\bar a)\right|^3}d\bar a-\frac{1}{4\pi}\frac{K_0(a)\mathbf{b}(a)}{2}2\ln(\frac{S_0/2}{s})\right)$$

References

CALLEGARI,A.J. & TING,L. 1978 Motion of a curved vortex filament with decaying vortical core and axial velocity, *SIAM J. Appl. Math.* **35** (1), 148-175

FRANCOIS,C. 1981 Les méthodes de perturbation en mécanique. ENSTA. Paris, 98-104

KLEIN,R. & KNIO,O.M. 1995 Asymptotic vorticity structure and numerical simulation of slender vortex filaments, *J. Fluid Mech.* **284**, 257-321

FUKUMOTO,Y. & MIYAZAKI,T. 1991 Three dimensional distortions of a vortex filament with axial velocity, *J. Fluid Mech.* **222**, 369-416