The different equations of motion of the central line of a slender Vortex Filament

&

Their use to the study of perturbed Vortices

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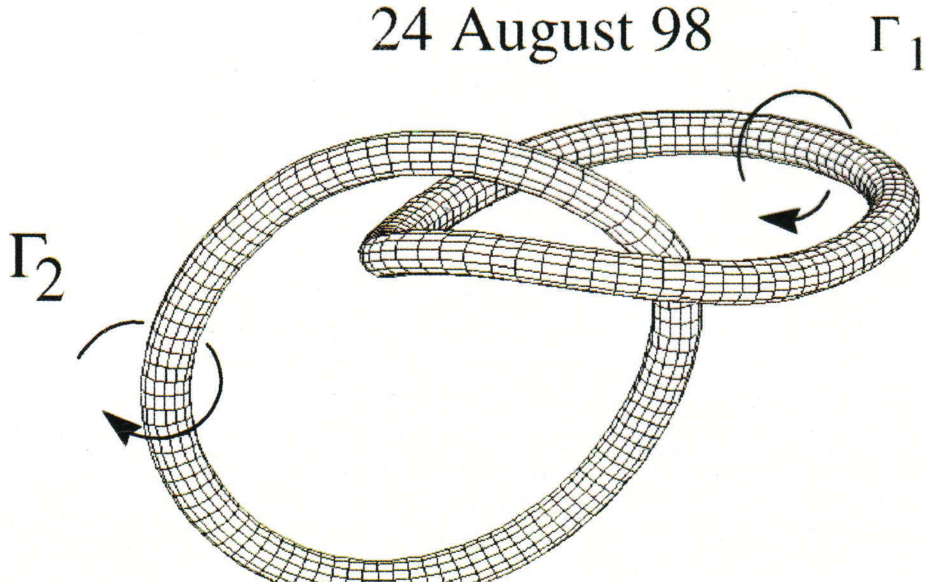


- References

- *Study of* **circular** *Rings:*

  Kelvin (1867) , Hicks(1884), Dyson(1893), Lamb(1906), Saffman (1970), Fraenkel (70) Fukumoto (1997)

- *Study of* **non circular** *Rings*
  - *Asymptotics Methods*:

    Levi-Civita (1932), Widnall (1971),

    Moore (1972), **Ting(1978)**, Fukumoto(1991),

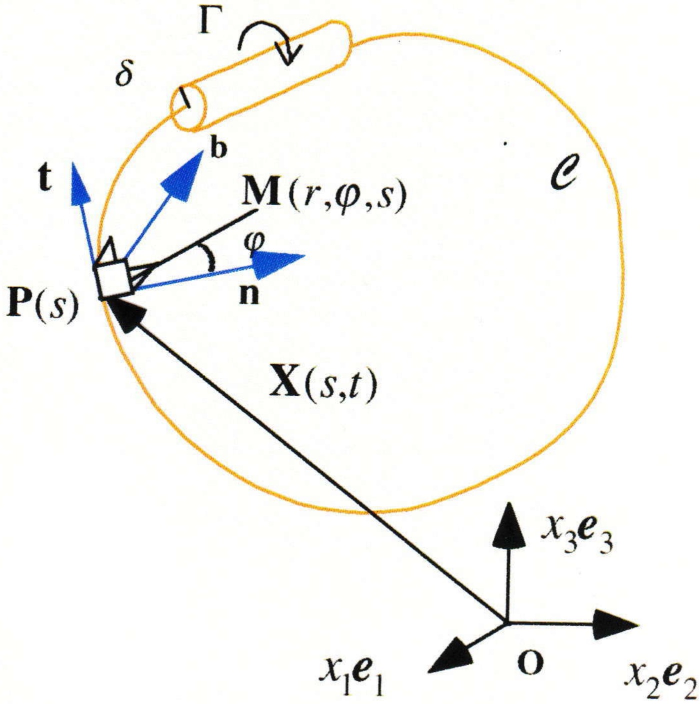
    Klein(1991,1995)

  - *Cut-off Methods : (ad-hoc regularization)*

    JJ.Thomson 1883, Hama-Burger 62, Crow 70 Levy 28, Rosenhead 30, Parks 70, Saffman 72-92,...

Contents

• Central line and space co-ordinates



- central line :

$\mathbf{X}(t,s,\varepsilon)$ and $(\mathbf{t},\mathbf{n},\mathbf{b})$

- Co-ordinates :

(r,φ,s) and $(\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{t})$

$$\mathbf{x}=\mathbf{X}(s,t)+r\mathbf{e}_r(\varphi,s,t)$$

-Biot & Savart :

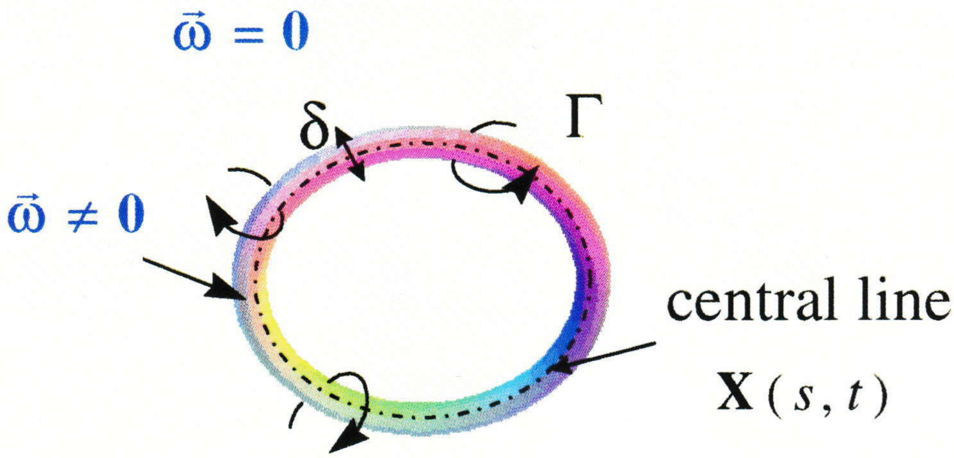
$$\mathbf{v}(r,\varphi,s,\varepsilon)=$$

$$\frac{1}{4\pi}\iiint\frac{\boldsymbol{\omega}(\bar{r}',\varphi',s')\times\left[(\mathbf{X}(s)+r\mathbf{e}_r(s,\varphi))-(\mathbf{X}(s')+\varepsilon\bar{r}'\mathbf{e}_r')\right]}{\left|(\mathbf{X}(s)+r\mathbf{e}_r(s,\varphi))-(\mathbf{X}(s')+\varepsilon\bar{r}'\mathbf{e}_r')\right|^3}h_3'\bar{r}'\,d\bar{r}'\,d\varphi'\,ds'$$

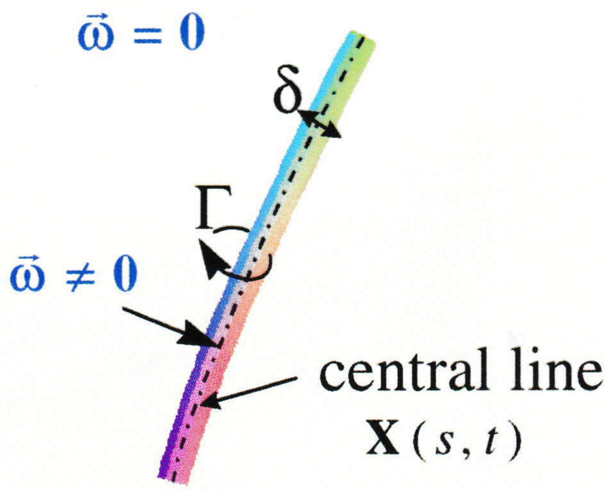
$$h_3'=\sigma(s',t)\ \ (1-K(s',t)\varepsilon\bar{r}'\cos(\varphi'))$$

• Slender Rings and Filaments :

Vortex Ring



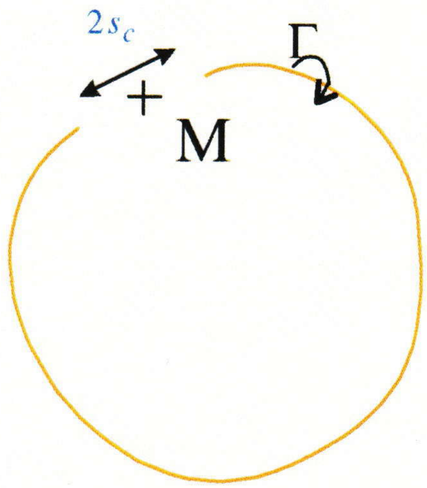
Vortex Filament



Problem : Motion? Interactions?
Stability ?

2
The cut-off methods and their justifications

– Burger :



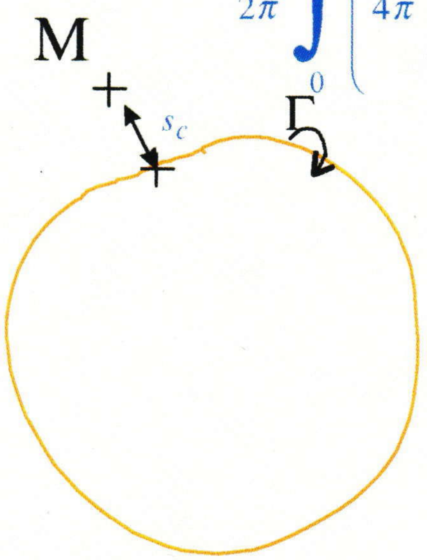
$$\dot{\mathbf{X}} = \frac{1}{4\pi}\int_I \left(\sigma(s',t)\,\frac{\mathbf{t}(s',t)\times(\mathbf{X}(s,t)-\mathbf{X}(s',t))}{\left|\mathbf{X}(s,t)-\mathbf{X}(s',t)\right|^3} \right) ds'$$

$$I = [0,2\pi[\setminus [s-s_c, s+s_c[$$

(JJ.Thomson 1883, Hama-Burger 62, Crow 70)

– JJ.Thomson 1883 :

$$\dot{\mathbf{X}} = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{4\pi}\int_0^{2\pi} \sigma(s',t)\,\frac{\mathbf{t}(s',t)\times(\mathbf{M}(s,t,s_c)-\mathbf{X}(s',t))}{\left|\mathbf{M}(s,t,s_c)-\mathbf{X}(s',t)\right|^3} ds' \right) d\varphi$$



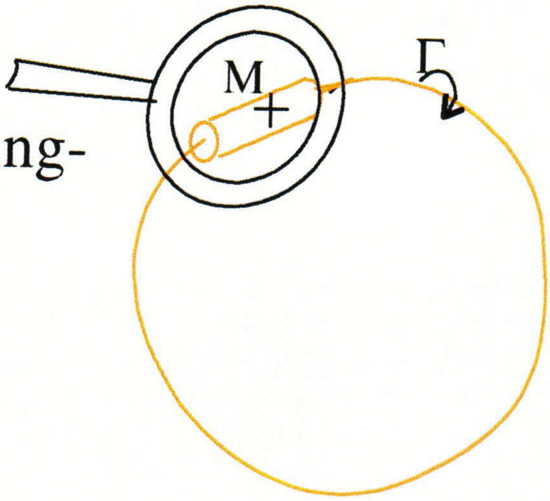
$$\mathbf{M}(s,t,s_c) = \mathbf{X}(s,t) + s_c\,\mathbf{e}_r(\varphi,s,t)$$

– Rosenhead :

$$\dot{\mathbf{X}} = \frac{1}{4\pi}\int_0^{2\pi} \sigma(s',t)\frac{\mathbf{t}(s',t)\times(\mathbf{X}(s,t)-\mathbf{X}(s',t))}{\left[\left(\mathbf{X}(s,t)-\mathbf{X}(s',t)\right)^2+s_c^2\right]^{3/2}}ds'$$

(Levy 28, Rosenhead 30, Parks 70, Saffman 72-92)

– Other :



magnifying-glass

$$\dot{\mathbf{X}} = \frac{1}{4\pi}\int_l \sigma(s',t)\frac{\mathbf{t}(s',t)\times\mathbf{MM}'}{\left|\mathbf{MM}'\right|^3} f\left(\frac{\left|\mathbf{MM}'\right|}{s_c}\right)ds'$$

$$\mathbf{MM}' = \mathbf{X}(s,t)-\mathbf{X}(s',t)$$

with $f(\chi^2)\longrightarrow 1$ when $\chi^2\longrightarrow\infty$

$$f(\chi^2) = \frac{\chi^3}{\left(\chi^2+1\right)^{3/2}}$$

VEM1

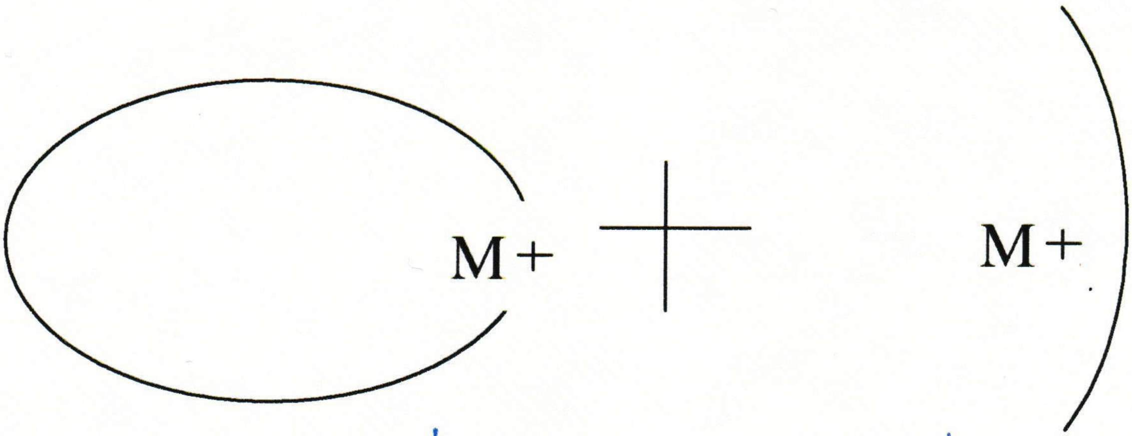
$$f(\chi^2) = \frac{-2\chi^2+\sqrt{\pi}\,\mathrm{erf}(\chi^2)e^{\chi^4}}{\sqrt{\pi}\,e^{\chi^4}}$$

VEM2

choice $s_c=\varepsilon$ Good choice ?

•Justification of cut-off methods

$s_c = 0 \Rightarrow$ singularity at $s' = s$



s_c=0 with s' fixed s_c=0 with $(s' - s)/l$ fixed

Identification : Cut-off $\leftrightarrow$ Callegari & Ting

$$s_c = \varepsilon \frac{1}{\sigma(\mathrm{s},\mathrm{t})} e^{N - C_V(t) - C_W(t)}$$

Nom de la méthode de cut-off	N
Burger	1-ln2
Thomson	1/2
Rosenhead	0
VEM1	0
VEM2	1-0.009122-ln2

s_c = the parameter of structure

- **The non closed Vortex filament**

– Asymptotic equation of motion :

$$\dot{\mathbf{X}}(s,t) = \mathbf{Q} + \frac{K(s,t)}{4\pi}\left[\ln\left[\frac{2}{\varepsilon} - 1\right] + C_v(t) + C_w(t)\right]\mathbf{b}$$

$$\mathbf{Q} = \mathbf{A} - (\mathbf{A}\cdot\mathbf{t})\,\mathbf{t}$$

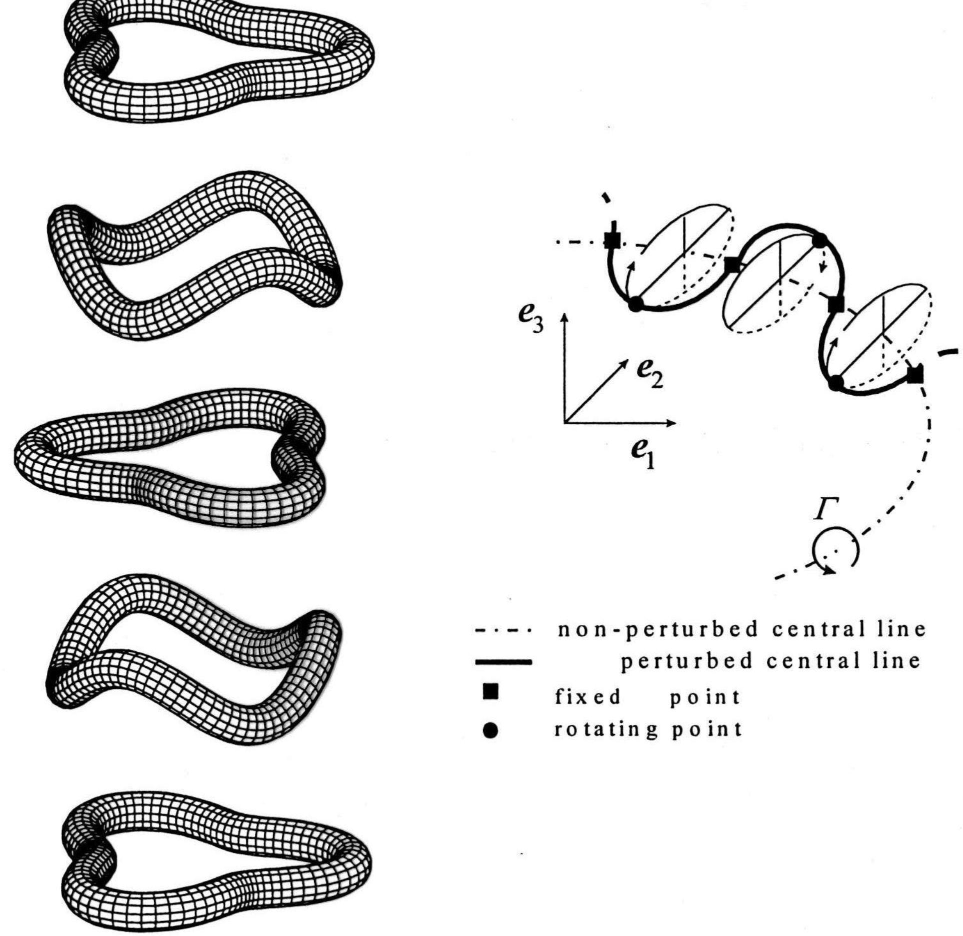
$$\mathbf{A} = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left[\frac{\mathbf{t}(s+s')\times\left(\mathbf{X}(s) - \mathbf{X}(s+s')\right)}{\left|\mathbf{X}(s) - \mathbf{X}(s+s')\right|^3} - H(1-|s'|)\frac{K(s)\mathbf{b}(s)}{|s'|}\right]ds'$$

– Cut-off Methods of Burgers :

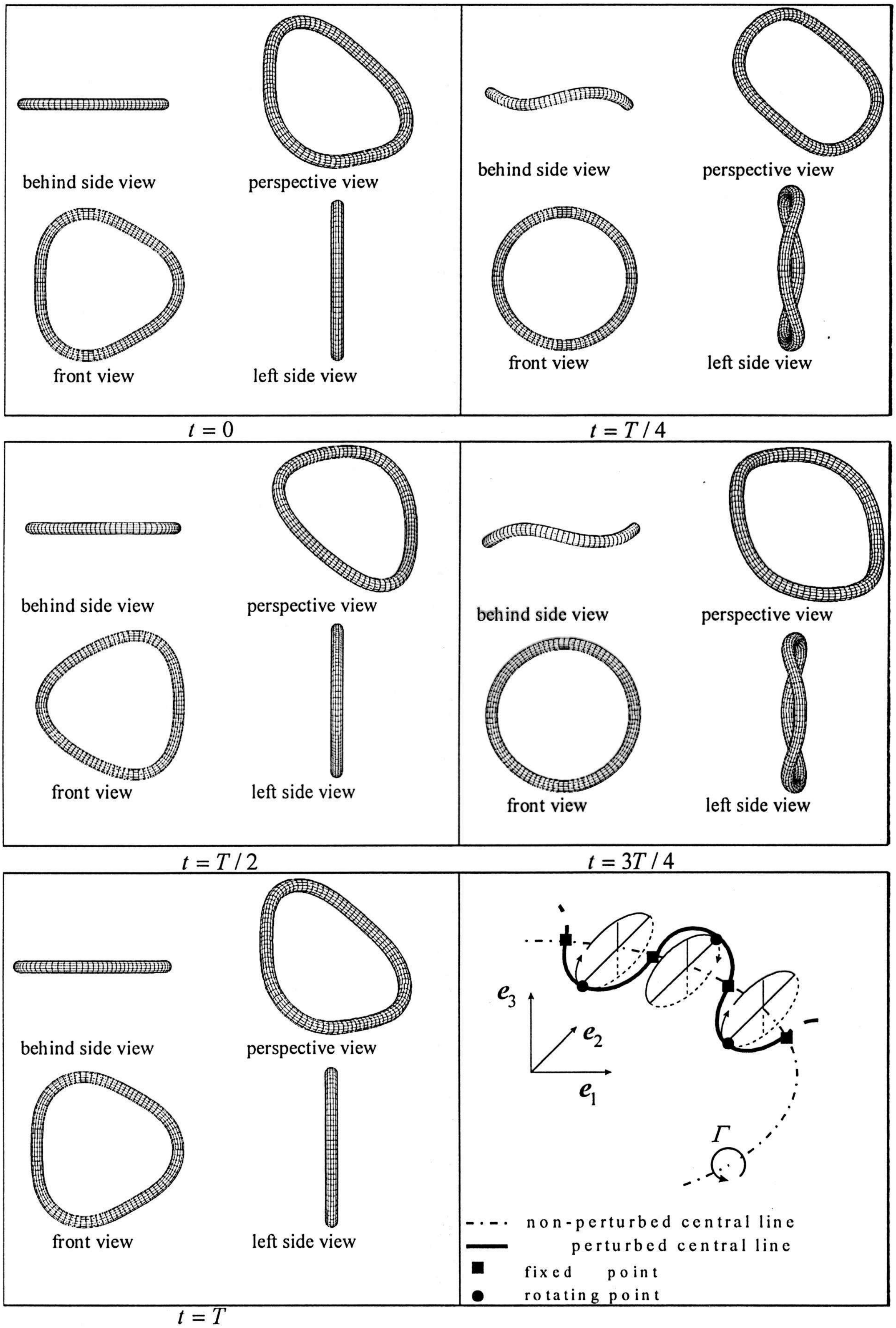
$$\dot{\mathbf{X}} = \frac{1}{4\pi}\int_I \frac{\mathbf{t}(s',t)\times\left(\mathbf{X}(s,t) - \mathbf{X}(s',t)\right)}{\left|\mathbf{X}(s,t) - \mathbf{X}(s',t)\right|^3}ds'$$

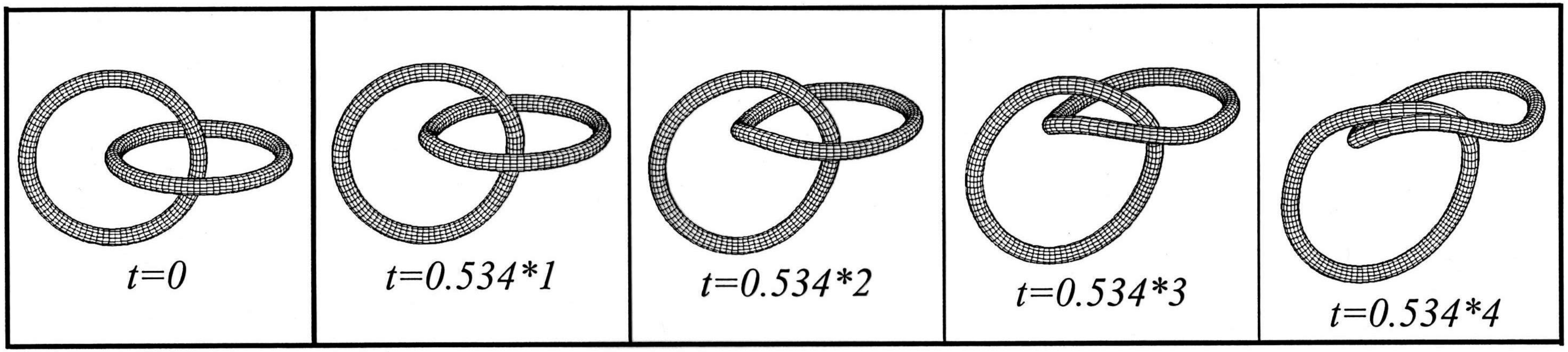
$$I = [-\infty, +\infty[\,\backslash\,[s - s_c, s + s_c[$$

3 Numerical Simulations

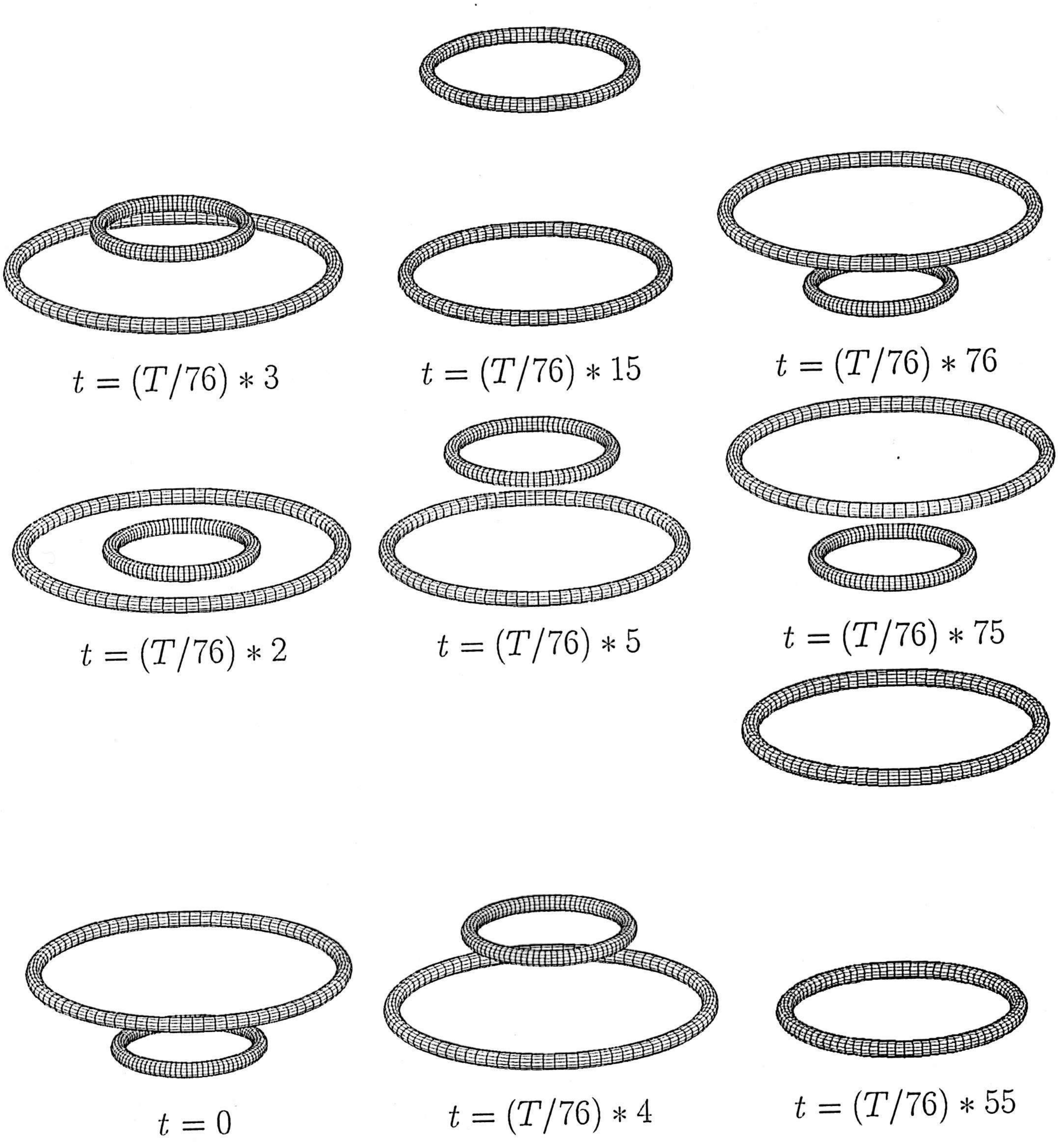


e_3
e_2
e_1
Γ
non-perturbed central line
perturbed central line
fixed point
rotating point

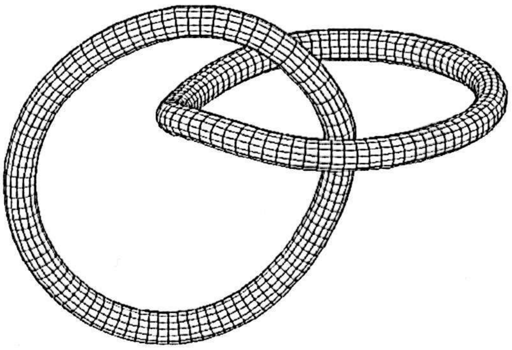

behind side view
perspective view
front view
left side view
t = 0
behind side view
perspective view
front view
left side view
t = T / 4
behind side view
perspective view
front view
left side view
t = T / 2
behind side view
perspective view
front view
left side view
t = 3T / 4
behind side view
perspective view
front view
left side view
t = T
e3
e2
e1
Γ
non-perturbed central line
perturbed central line
fixed point
rotating point



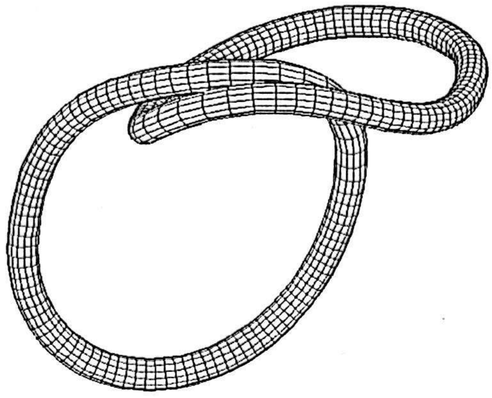
t=0
t=0.534*1
t=0.534*2
t=0.534*3
t=0.534*4



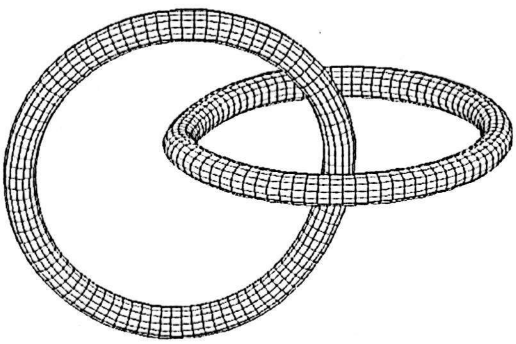
Leapfrogging of two vortex rings



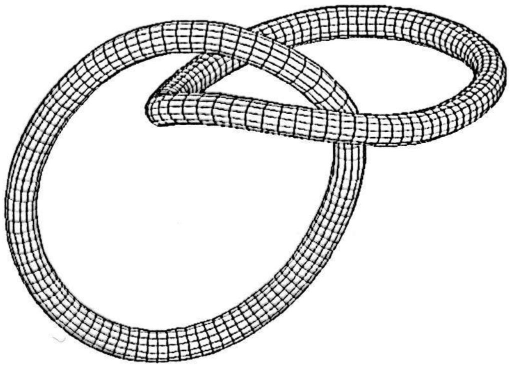
$t = 0.534 * 2$



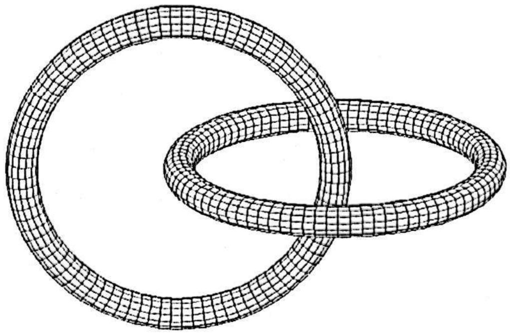
$t = 0.534 * 4$



$t = 0.534 * 1$



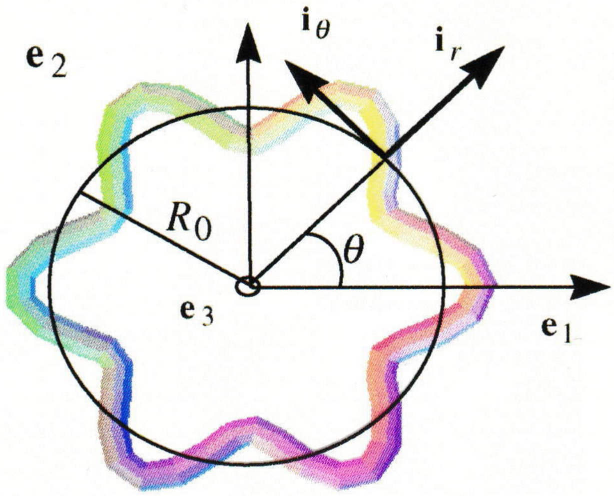
$t = 0.534 * 3$



$t = 0$

Evolution of two linked vortex rings

3 Oscillations of a perturbed circular ring



mode n=6

- Linear study of a circular vortex :

  Stability study :

  JJ.Thomson 1882- Widnall 73

  Our work :

  - We complet Widnall study :
    stable part with oscillations
  - Viscous case with Axial velocity componant

- **Oscillation Period of n mode:**

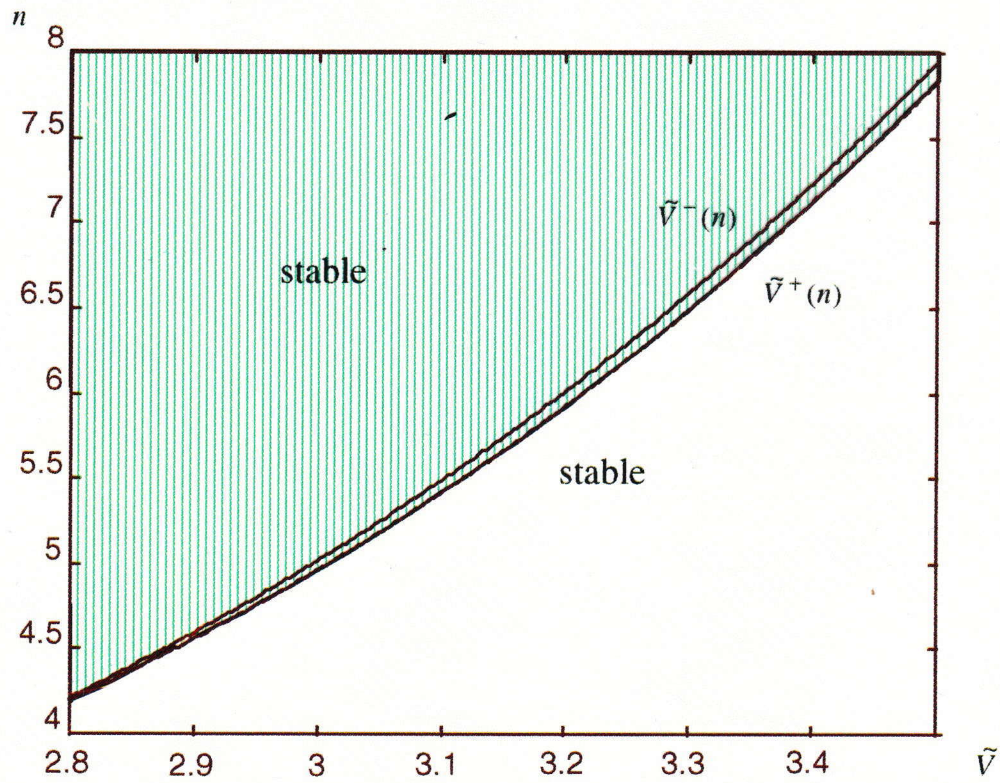
$$T = 8\pi^2 \Big/ \sqrt{\left[n^2\tilde{V} - g_{\xi}(n)\right]\left[(n^2-1)\tilde{V} + g_{\rho}(n)\right]}$$

$$\tilde{V} = \ln(8/\varepsilon) - 1 + C_v(t) - 2m_0^2 = \ln(4/s_c)$$

Parameter of structure

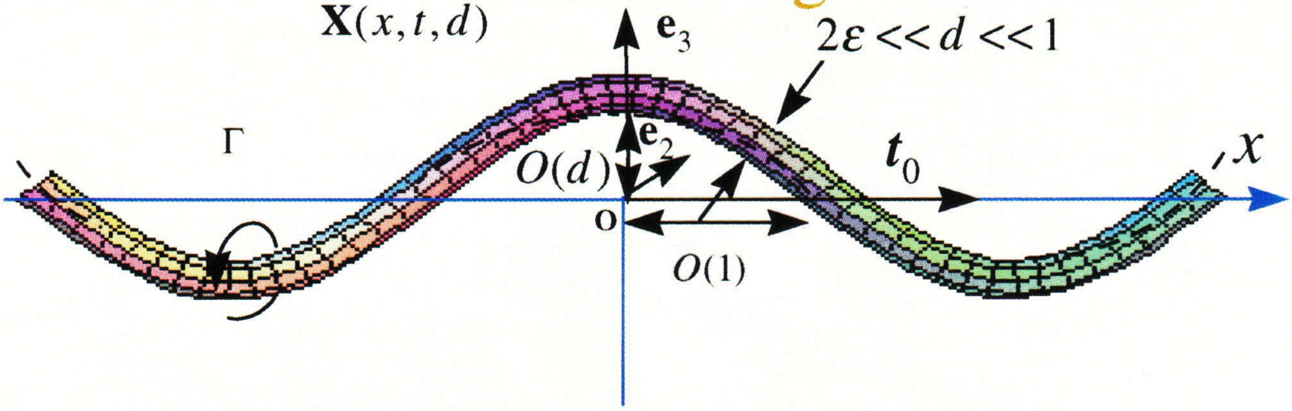
n	2	3	4
$g_{\xi}(n)$	6.66	20.8	43.123
$g_{\rho}(n)$	-7	- 21.33	- 43.8

- **Diagram of Stability**



5 The straight vortex filament

- Oscillation Period of a straight filament :



$$\boxed{\mathbf{X}(x,t,d) = x\mathbf{t}_0 + d\mathbf{X}^{(2)}(x,t) + o(d)}$$

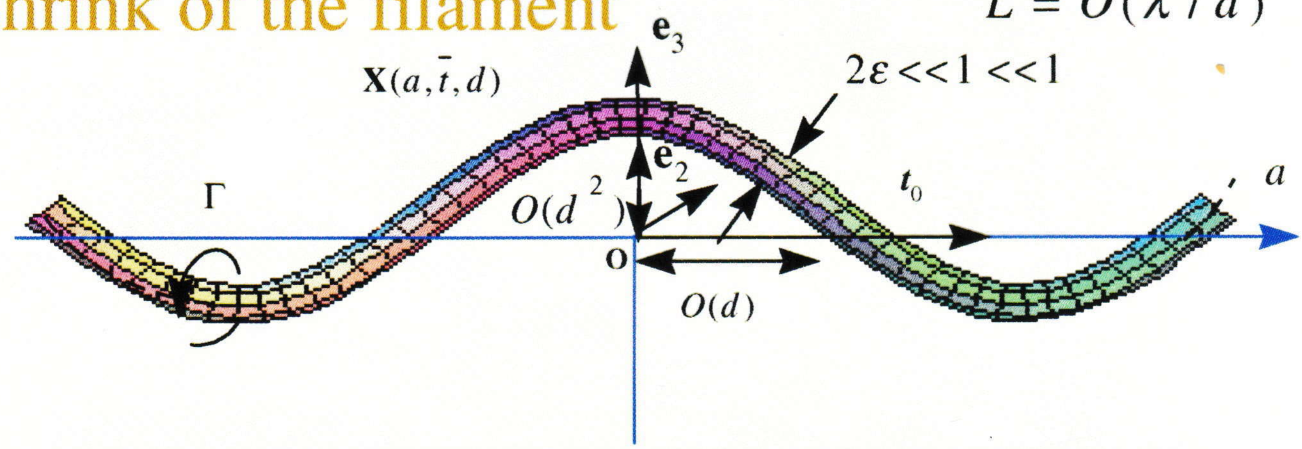
$\ln \varepsilon = O(1) \Rightarrow$ $$T = 8\pi^2 \Big/ \left| k^2 (\tilde{V} + \frac{1}{2} - \gamma - \ln k) \right|$$

$$\tilde{V} = \ln(8/\varepsilon) - 1 + C_v(t) - 2m_0^2 = \ln(4/s_c)$$

Typical length $L = O(\lambda)$

- Shrink of the filament

$L = O(\lambda / d)$



$$\boxed{\mathbf{X}(a,\bar{t},d) = a\mathbf{t}_0 + d^2\mathbf{X}^{(2)}(x = a/d, t = \bar{t}/d^2) + o(d^2)}$$

Klein & Majda filament (91)

Simplified equation of motion

$$\frac{\partial \mathbf{X}^{(2)}}{\partial \tau}(x,\tau) = \mathbf{t}_0 \times \mathbf{X}^{(2)}_{xx}(x,\tau) - \tilde{\alpha}\mathbf{t}_0 \times \mathbf{J}(x,\tau)$$

$$\mathbf{J}(x,\tau) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left[\frac{\left(\mathbf{X}^{(2)}(x+h,\tau) - \mathbf{X}^{(2)}(x,\tau)\right) - h\frac{\partial \mathbf{X}^{(2)}}{\partial x}(x+h,\tau)}{|h|^3} + H(1-|h|)\frac{\frac{\partial \mathbf{X}^{(2)}}{\partial x \partial x\cdot}(x,\tau)}{2|h|}\right]dh$$

$\tau = t/\tilde{\alpha}$ $\qquad$ $\tilde{\alpha} = 4\pi/\left(\left[\ln\frac{2}{\varepsilon} - 1\right] + C^*\right) = -\frac{1}{4\pi \ln s_c}$ $\qquad$ Parameter of structure

Transformation of Hasimoto $\qquad \psi = K\exp(i\int_0^s T\,ds)$

$$\Rightarrow \quad \frac{1}{i}\psi_\tau = \psi_{xx} + d^2(\frac{1}{2}|\psi|^2\psi) - \tilde{\alpha}\,(J(\psi) + O(d))$$

equation of Schrödinger

Local Induction Regime (LIA)

$$\frac{1}{i}\psi_\tau = \psi_{xx} + d^2\frac{1}{2}|\psi|^2\psi$$

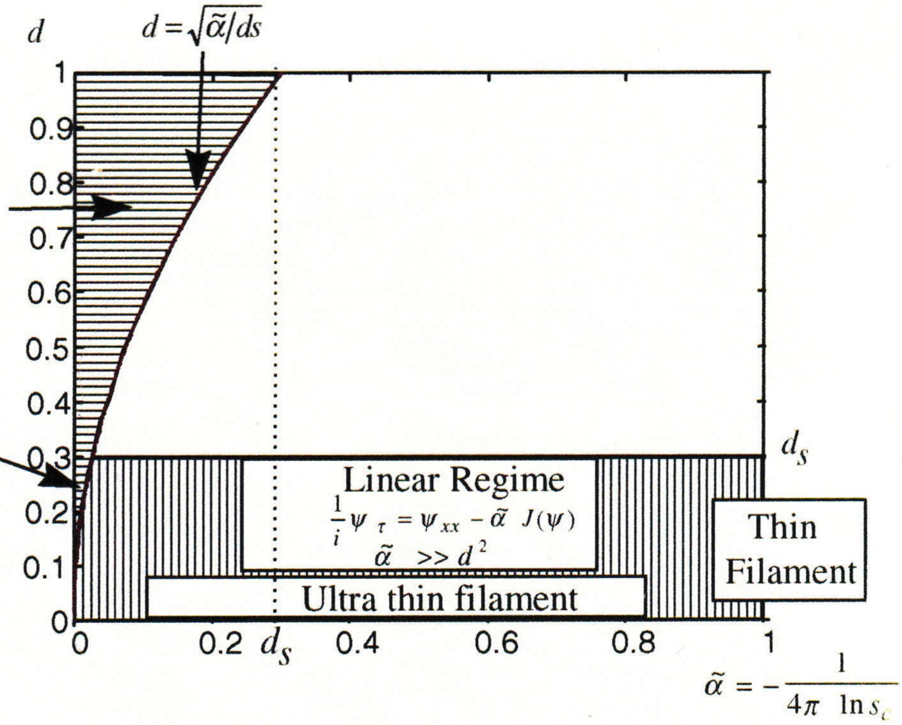
$$\tilde{\alpha} \ll d^2$$

Klein & Majda Regime

$$\frac{1}{i}\psi_\tau = \psi_{xx} + d^2(\frac{1}{2}|\psi|^2\psi - \kappa J(\psi))$$

$$\tilde{\alpha} = O(d^2)$$

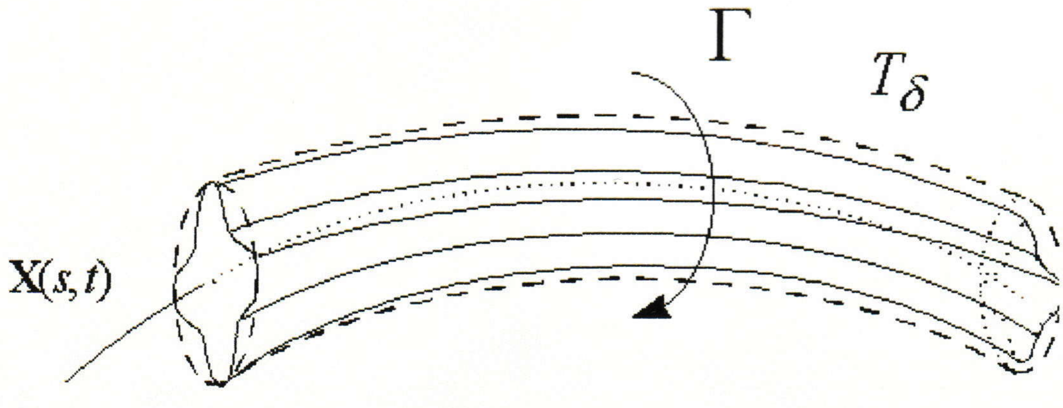
$$\kappa = \tilde{\alpha}/d^2 = O(1)$$



Perspectives

- Helicity and its evolution :
  Moffatt (1969), Ricca (1992)

- Axial Variation or Non Axisymetric core



Lundgren (1989-1982)

Symbolic calculus

- Dipolar ring