Expansion of the Biot-Savart law applied to a curved slender vortex filament with axial variation

Daniel Margerit[1]

[1] Mathematics Institute, University of Warwick

Invited by Stephen J. Cowley for a talk at DAMTP, University of Cambridge, 2 Fev. 2000 (Refactored in Mai 2024)

Abstract

We will show how to obtain the two first orders of the expansion of the potential flow induced by a curved vortex line. The expansion is done on the distance r to the line and is performed by using the method of matched asymptotic expansion of singular integral. The order $O(r)$ of this expansion will be given with both its local and global integral parts. This method will also be used to give the inner expansion of the flow induced by a slender vortex with the slenderness ϵ as the expansion parameter and an application will be given for a circular vortex ring with axial core structure variation. Finally we will summarise the derivation of the equation of motion of the central line [Callegari and Ting SIAM J.Appl. Math. 35(1), 1978], give its generalisation at next order and speak about its generalisation to a curved filament with axial variation.

Outline

Velocity and vorticity equations

- Domaine of study: Incompressive laminar flows with vorticity and $Re \gg 1$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v} = -\nabla\frac{p}{\rho} + \nu\Delta\mathbf{v}$$

$$\nabla \cdot \mathbf{v} = 0$$

- The field of Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{v}$$

$$\frac{\partial \omega}{\partial t} + \mathbf{v}\nabla\omega = \omega\nabla\mathbf{v} + \nu\Delta\omega$$

- Biot and Savart

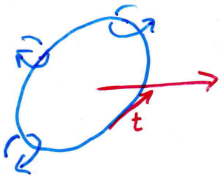
$$\mathbf{v}(\mathbf{x}) = \nabla\varphi(\mathbf{x}) + \frac{1}{4\pi}\iiint \frac{\omega(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}',$$

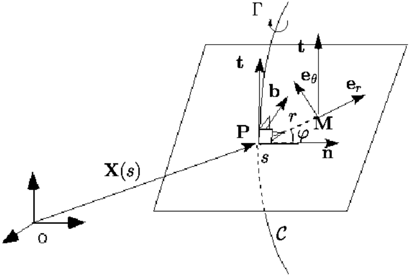
Background velocity $\nabla\varphi(\mathbf{x})$ + Induced velocity

Outline

The curved vortex line

The curved filament without thickness $\omega = \Gamma \delta_{\mathcal{C}} \mathbf{t}$





The velocity expansion near the curved line

If $\boldsymbol{\omega} = \Gamma \delta_{\mathcal{C}} \mathbf{t}$ then

$$\mathbf{v}(\mathbf{x}) = \frac{\Gamma}{4\pi} \int_{\mathcal{C}} \frac{\mathbf{t}(a') \times (\mathbf{x} - \mathbf{X}(a'))}{|\mathbf{x} - \mathbf{X}(a')|^3} da'$$

$$\mathbf{v}(r \to 0) = \frac{\Gamma}{4\pi}\mathbf{e}_\theta + \frac{\Gamma}{4\pi} K \ln \frac{\mathcal{L}}{r} \mathbf{b} + \mathbf{Q}_f + O(r)$$

The leading order expansion near the curved line

Dimensionless : length $L = O(1/K)$ velocity Γ/L

$$\mathbf{v}(r \to 0, \varphi, a) = \frac{1}{2\pi r}\mathbf{e}_\theta + \frac{K}{4\pi}\cos\varphi\mathbf{e}_\theta + \mathbf{A} + \frac{K}{4\pi}\left[\ln\frac{S}{r} - 1\right]\mathbf{b} + O(r)$$

where

$$\mathbf{A}(a) = \frac{1}{4\pi}\int_{-S/2}^{+S/2}[\frac{\mathbf{t}(a+a^*)\times\mathbf{d}}{|\mathbf{d}|^3} - \frac{K(a)\mathbf{b}(a)}{2|a^*|}]da^*,$$

with $\mathbf{d} = \mathbf{X}(a) - \mathbf{X}(a+a^*)$
Result :

$$\mathcal{L} = S$$

$$\mathbf{Q}_f = \frac{\Gamma K}{4\pi}(\cos\varphi\mathbf{e}_\theta - \mathbf{b}) + \Gamma\mathbf{A}$$

The first order expansion near the curved line

Dimensionless : length $L = O(1/K)$ velocity Γ/L

$$\mathbf{v}(r \to 0, \varphi, a) = \frac{1}{2\pi r}\mathbf{e}_\theta + \frac{K}{4\pi}\cos\varphi \mathbf{e}_\theta + \mathbf{A} + \frac{K}{4\pi}\left[\ln\frac{S}{r} - 1\right]\mathbf{b}$$
$$+ r\mathbf{I} + \frac{1}{4\pi}\left(\mathbf{B} - 3\mathbf{C} - \frac{4}{S^2}\mathbf{e}_\theta\right) r + O(r^2 \ln r),$$

where[1]

$$\begin{aligned}\mathbf{I} \quad = \quad & \frac{3}{16}\frac{K^2}{\pi}\left[(\mathbf{e}_r \sin 2\varphi + \mathbf{e}_\theta \cos 2\varphi)\left(\ln\frac{S}{r} - \frac{4}{3}\right)\right] \\ & + \frac{3}{16}\frac{K^2}{\pi}\left[\frac{1}{2}\mathbf{e}_\theta \cos 2\varphi + \frac{1}{18}\mathbf{e}_\theta\right] \\ & + \frac{1}{4\pi}(K_a \sin\varphi - KT\cos\varphi)\left[\ln\frac{S}{r} - 1\right]\mathbf{t}.\end{aligned}$$

[1] Y. Fukumoto and T. Miyazaki. "Three dimensional distortions of a vortex filament with axial velocity". In: *J. Fluid Mech.* 222 (1991), pp. 369–416.

The first order expansion near the curved line

$$\mathbf{A}(a) = \frac{1}{4\pi}\int_{-S/2}^{+S/2}[\frac{\mathbf{t}(a+a^*)\times\mathbf{d}}{|\mathbf{d}|^3} - \frac{K(a)\mathbf{b}(a)}{2|a^*|}]da^*,$$

$$\mathbf{B}(\varphi, a) = \mathbf{e}_r(\varphi, a)\times\int_{-S/2}^{+S/2}[-\frac{\mathbf{t}(a+a^*)}{|\mathbf{d}|^3} - f_b(a, a^*)]da^*,$$

$$\mathbf{C}(\varphi, a) = \int_{-S/2}^{+S/2}[\frac{\mathbf{e}_r(\varphi, a)\cdot\mathbf{d}}{|\mathbf{d}|^5}[\mathbf{t}(a+a^*)\times\mathbf{d}] - f_c(a, a^*)]da^*,$$

with $\mathbf{d} = \mathbf{X}(a) - \mathbf{X}(a+a^*)$ and[2]

$$\begin{aligned} f_b(a, a^*) &= -\frac{1}{|a^*|^3}[\mathbf{t}(a) + K(a)\mathbf{n}(a)a^*] \\ &\quad -\frac{1}{|a^*|^3}[\frac{a^{*2}}{2}[K_a(a)\mathbf{n}(a) + K(a)T(a)\mathbf{b}(a) - \frac{3}{4}K^2(a)\mathbf{t}(a)]] \\ f_c(a, a^*) &= -\frac{K^2(a)\mathbf{b}(a)\cos(\varphi)}{4|a^*|}. \end{aligned}$$

[2] D. Margerit and J-P Brancher. "Asymptotic expansions of the Biot-Savart law for a slender vortex with core variation". In: *Journal of Engineering Mathematics* 40.3 (2001), pp. 297–313.

The proof : the MAESI Method

MAESI = Matched Asymptotic Expansion of Singular Integral[3]

1. Change of variable $a^* = a' - a$ is done :

$$\begin{aligned} \mathbf{v}(r, \varphi, a) &= \frac{1}{4\pi} \int_{\mathcal{C}} \frac{\mathbf{t}(a') \times (\mathbf{x} - \mathbf{X}(a'))}{|\mathbf{x} - \mathbf{X}(a')|^3} da' \\ &= \frac{1}{4\pi} \int_{-S/2}^{+S/2} \mathbf{K}(r, \varphi, a, a^*) da^*, \end{aligned}$$

2. Small intermediate parameter η with $r \ll \eta \ll 1$

$$\mathbf{v}(r, \varphi, a) = \mathbf{Ex} + \mathbf{In}$$

$$\begin{aligned} \mathbf{Ex} &= \frac{1}{4\pi} \int_{-S/2}^{-\eta} \mathbf{K} da^* + \frac{1}{4\pi} \int_{\eta}^{+S/2} \mathbf{K} da^* \\ \mathbf{In} &= \frac{1}{4\pi} \int_{-\eta}^{\eta} \mathbf{K} da^* \end{aligned}$$

[3] C. François. *Les méthodes de perturbation en mécanique*. 98-104. Paris: ENSTA, 1981.

The proof : the MAESI Method

3. Stretched variable $\bar{a} = a^*/r$ introduced

$$\mathbf{In} = \frac{1}{4\pi} r \int_{-\eta/r}^{\eta/r} \tilde{\mathbf{K}} d\bar{a}$$

$$\tilde{\mathbf{K}}(r, \varphi, a, \bar{a}) = \mathbf{K}(r, \varphi, a, r\bar{a})$$

4. Outer Part : expansion with r of **Ex**

- expansion of **K** with r and integrate
- expansion with $\eta \ll 1$

Example :

$$\int_{\eta}^{+S/2} \frac{\mathbf{t}(a+a^*) \times \mathbf{d}}{|\mathbf{d}|^3} da^* = \int_{\eta}^{+S/2} \frac{K(a)\mathbf{b}(a)}{2|a^*|} da^* + \int_{\eta}^{+S/2} [\frac{\mathbf{t}(a+a^*) \times \mathbf{d}}{|\mathbf{d}|^3} - \frac{K(a)\mathbf{b}(a)}{2|a^*|}] da^*$$

because

$$\mathbf{t}(a+a^*) \times \mathbf{d}/|\mathbf{d}|^3 = K(a)\mathbf{b}(a)/2|a^*| + O(1)$$

The proof : the MAESI Method

5. Inner part : expansion with r of **In**
   - expansion of $\tilde{\mathbf{K}}$ with r and integrate
   - expansion of **In** with $\eta/r \gg 1$
6. Addition of **Ex** and **In** :

η disappears

Outline

The flow induced by a curved slender vortex

$$\mathbf{v}(\mathbf{x}) = \frac{1}{4\pi} \iiint \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}',$$

Dimensionless : length $L = O(1/K)$ velocity Γ/L

- Outer limit : $\epsilon \to 0$ with r fixed: $\mathbf{v}^{out} = \mathbf{v}^{out(0)} + \epsilon \mathbf{v}^{out(1)} + O(\epsilon^2)$
- Inner limit : $\epsilon \to 0$ with $\bar{r} = r/\epsilon$ fixed: $\mathbf{v}^{inn} = \epsilon^{-1}\mathbf{v}^{inn(0)} + \mathbf{v}^{inn(1)} + O(\epsilon)$

$$\mathbf{v}(r, \varphi, s, t, \epsilon) = \frac{1}{4\pi} \iiint \frac{\epsilon^2 \boldsymbol{\omega}(\bar{r}', \varphi', a', \epsilon) \times [\mathbf{X} + r\mathbf{e}_r - (\mathbf{X}' + \epsilon \bar{r}' \mathbf{e}_r')]}{|\mathbf{X} + r\mathbf{e}_r - (\mathbf{X}' + \epsilon \bar{r}' \mathbf{e}_r')|^3} h_3' \bar{r}' d\bar{r}' d\varphi' da',$$

where $h_3' = (1 - K(a')\epsilon \bar{r}' \cos(\varphi'))$.
Problem : Velocity induced by $\boldsymbol{\omega}(\mathbf{x}) = \boldsymbol{\omega}(\bar{r}, \varphi, a, \epsilon) = \epsilon^{-2}\boldsymbol{\omega}^{(0)}(\bar{r}, \varphi, a)$

The flow induced by a curved slender vortex

- $\mathbf{v}^{out(0)}$: line vortex, $\mathbf{v}^{out(1)} \neq 0$
- MAESI[4] method in ϵ :

$$\mathbf{v}^{inn(0)} = -\frac{1}{2\pi} \iint \mathbf{g} \bar{r}' d\bar{r}' d\varphi'$$

where

$$\begin{aligned} \mathbf{g} &= \boldsymbol{\omega}^{(0)}(\bar{r}', \varphi', a) \times \left[\bar{r}' \mathbf{e}_r(\varphi', a) - \bar{r} \mathbf{e}_r(\varphi, a) \right] / k^2 \\ k^2 &= \bar{r}^2 + \bar{r}'^2 - 2\bar{r}\bar{r}' \cos(\varphi - \varphi') \end{aligned}$$

[4] Margerit and Brancher, "Asymptotic expansions of the Biot-Savart law for a slender vortex with core variation".

The flow induced by a curved slender vortex

$$
\begin{aligned}
\mathbf{v}^{inn(1)} =& \mathbf{A} + \frac{K}{4\pi}[\ln\frac{S}{\epsilon} - 1]\mathbf{b} \\
& - \frac{1}{4\pi}\iint \boldsymbol{\omega}_a^{(0)}(\bar{r}',\varphi',a) \times \mathbf{t}(a)\ln\frac{1}{k^2}\bar{r}'d\bar{r}'d\varphi' \\
& - \frac{K(a)}{8\pi}\iint \boldsymbol{\omega}^{(0)}(\bar{r}',\varphi',a) \times \mathbf{n}(a)\ln\frac{1}{k^2}\bar{r}'d\bar{r}'d\varphi' \\
& - \frac{K(a)\bar{r}\cos(\varphi)}{4\pi}\iint \mathbf{g}\bar{r}'d\bar{r}'d\varphi' + \frac{K(a)}{4\pi}\iint \mathbf{g}\cos\varphi'\bar{r}'^2 d\bar{r}'d\varphi', \\
& - \frac{T(a)\bar{r}}{2\pi}\iint \frac{\boldsymbol{\omega}^{(0)}(\bar{r}',\varphi',a)\times\mathbf{t}(a)}{k^2}\sin(\varphi-\varphi')\bar{r}'^2 d\bar{r}'d\varphi'
\end{aligned}
$$

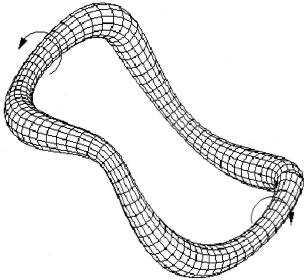
where

$$
\begin{aligned}
\mathbf{g} &= \boldsymbol{\omega}^{(0)}(\bar{r}',\varphi',a) \times \left[\bar{r}'\mathbf{e}_r(\varphi',a) - \bar{r}\mathbf{e}_r(\varphi,a)\right]/k^2 \\
k^2 &= \bar{r}^2 + \bar{r}'^2 - 2\bar{r}\bar{r}'\cos(\varphi-\varphi')
\end{aligned}
$$

Application to a slender vortex with axial variation

- Chosen vorticity[5]

$$\boldsymbol{\omega} = \frac{1}{\epsilon^2}\left[\frac{1}{\pi \bar{r}_0^2(a)}\mathbf{t} + \epsilon \frac{\bar{r}_0'(a)}{\pi \bar{r}_0^3(a)}\frac{\bar{r}}{h_3}\mathbf{e}_r + \frac{g(a,\bar{r})}{h_3}\mathbf{e}_\theta\right] H\left(1 - \frac{\bar{r}}{\bar{r}_0(a)}\right)$$



$\mathrm{div}\boldsymbol{\omega} = 0$ and $\boldsymbol{\omega}\cdot\mathbf{N} = 0$ on $\bar{r} = \bar{r}_0(a)$

- Decomposition $\mathbf{v} = \mathbf{v}(\bar{r} = 0, a) + \mathbf{V}$

[5] Margerit and Brancher, "Asymptotic expansions of the Biot-Savart law for a slender vortex with core variation".

Application to a slender vortex with axial variation

- Induced velocity : $(g(a,\bar{r})=0)$

$$\begin{aligned}\mathbf{V}^{inn(0)} &= \frac{\bar{r}}{2\pi\bar{r}_0^2}\mathbf{e}_\theta \text{ if } \bar{r}<\bar{r}_0(s)\\ &= \frac{1}{2\pi\bar{r}}\mathbf{e}_\theta \text{ if } \bar{r}>\bar{r}_0(s),\end{aligned}$$

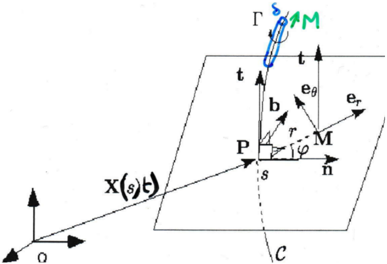
$$\begin{aligned}\mathbf{V}^{inn(1)} &= -\left(\frac{\bar{r}}{\bar{r}_0}\right)^2\frac{K}{16\pi}\left[3\sin\varphi\mathbf{e}_r+\cos\varphi\mathbf{e}_\theta\right] \qquad \text{if } \bar{r}<\bar{r}_0(s)\\ &= -\frac{K}{16\pi}\left(\left[-\left(\frac{\bar{r}_0}{\bar{r}}\right)^2+4+4\ln\frac{\bar{r}}{\bar{r}_0}\right]\sin\varphi\mathbf{e}_r\right.\\ &\qquad \left.+\left[\left(\frac{\bar{r}_0}{\bar{r}}\right)^2+4\ln\frac{\bar{r}}{\bar{r}_0}\right]\cos\varphi\mathbf{e}_\theta\right) \text{ if } \bar{r}>\bar{r}_0(s)\end{aligned}$$

and

$$\mathbf{v}(\bar{r}=0,a)=\mathbf{A}+\frac{K\mathbf{b}}{4\pi}\left[\ln\frac{S}{(\epsilon\bar{r}_0)}\right]$$

Outline

The curved slender filament



$$\mathbf{v} = \dot{\mathbf{X}}(s,t) + \mathbf{V}$$

$$\mathbf{V} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{t}$$

Dimensionless :
length $L = O(1/K) \gg l = O(\delta)$, $l/L = \epsilon$, velocity Γ/L

$$Re = \frac{\Gamma}{\nu} = \frac{1}{\alpha^2\epsilon^2} \gg 1 \qquad Sw = \frac{\Gamma\delta}{M} = \frac{\epsilon}{M} = \frac{1}{m} + O(\epsilon)$$

Equation of Callegari and Ting[6]

$$\dot{\mathbf{X}}(s,t) = \mathbf{Q} + \frac{K(s,t)}{4\pi}\left[\ln(S/\epsilon) - 1 + C_v(t) + C_w(t)\right]\mathbf{b}(s,t)$$

$$\begin{aligned}
\text{with } \mathbf{Q} &= \mathbf{A}(s,t) - [\mathbf{A}(s,t)\cdot\mathbf{t}(s,t)]\,\mathbf{t}(s,t) \\
\mathbf{A}(s,t) &= \frac{1}{4\pi}\int_{-\pi}^{+\pi} ds'\ \sigma' \left[\frac{\mathbf{t}(s+s',t)\times\mathbf{d}}{|\mathbf{d}|^3} - \frac{K(s,t)\mathbf{b}(s,t)}{2\,|\lambda(s,s',t)|}\right] \\
\mathbf{d} &= \mathbf{X}(s,t) - \mathbf{X}(s+s',t)
\end{aligned}$$

[6] A.J. Callegari and L. Ting. “Motion of a curved vortex filament with decaying vortical core and axial velocity”. In: *SIAM J. Appl. Math.* 35.1 (1978), pp. 148–175.

Cut-off equations

$$\partial \mathbf{X}/\partial t = \frac{1}{4\pi}\int_I \sigma(s',t)\frac{\mathbf{t}(s',t)\times(\mathbf{X}(s,t)-\mathbf{X}(s',t))}{|\mathbf{X}(s,t)-\mathbf{X}(s',t)|^3}ds'$$

where $I = [0,2\pi[\setminus[s-s_c, s+s_c[$ s_c : cut-off length
Singular Integral for the small parameter s_c
MAESI + Comparison[7]

$$s_c(s,t) = \epsilon \exp\left(1 - \ln 2 - C_v(t) - C_w(t)\right)/\sigma(s,t)$$

Regularization methods

[7] D. Margerit. "Mouvement et dynamique des filaments et des anneaux tourbillons de faible Ã©paisseur". PhD thesis. Institut National Polytechnique de Lorraine, 1997.

Outline

1. Equations on curvilinear coordinates (r, φ, s)

$$\begin{aligned} \mathbf{v}(r,\varphi,s,t,\epsilon) &= \dot{\mathbf{X}}(s,t,\epsilon) + \mathbf{V}(r,\varphi,s,t,\epsilon) \\ \mathbf{V} &= u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{t} \end{aligned}$$

- Biot and Savart in local coordinates
- Continuity in local coordinates

$$(urh_3)_r + (h_3 v)_\varphi + rw_s - Trw_\varphi = -r\dot{\mathbf{X}}_s \cdot \mathbf{t}$$

where $h_3 = \sigma\,(1 - rK(s)\cos(\varphi))$ and $\sigma = |\mathbf{X}_s|$

- Navier-Stokes in local coordinates

$$\begin{aligned} \mathbf{a} &= -\nabla p + \nu\Delta\mathbf{V} + \frac{\nu}{h_3}\left(\frac{1}{h_3}\dot{\mathbf{X}}_s\right)_s \\ \mathbf{a} &= \left(\frac{\partial \mathbf{V}}{\partial t}\right)_{r,\varphi,s} + (\mathbf{V} - r\dot{\mathbf{e}_r})\cdot\nabla\mathbf{V} + \frac{\dot{\mathbf{X}}_s}{h_3}\left(w - r\dot{\mathbf{e}_r}\cdot\mathbf{t}\right) \end{aligned}$$

$$\left(\frac{\partial \mathbf{V}}{\partial t}\right)_{r,\varphi,s} = \frac{\partial u}{\partial t}\mathbf{e}_r + u\dot{\mathbf{e}_r} + \dots$$

2. Expansions and Matching

$$\mathbf{X} \quad = \quad \mathbf{X}^{(0)}(s,t) \quad + \quad \epsilon \mathbf{X}^{(1)}(s,t) \quad + \ldots$$

- Outer limit: $\epsilon \to 0$ with r fixed:

$$\mathbf{v}^{out} = \mathbf{v}^{out(0)} + \epsilon \mathbf{v}^{out(1)} + O(\epsilon^2)$$

  - Expansion in the Biot Savart law
  - Matching law: expansion at $r = 0$ and replace $r = \bar{r}\epsilon$
    $\Rightarrow$ Boundary Conditions at $\bar{r} = \infty$
    Only $\cos(\varphi)$ and $\sin(\varphi)$ parts depend on the unknown $\mathbf{X}$!
- Inner limit: $\epsilon \to 0$ with $\bar{r} = r/\epsilon$ fixed:

$$\begin{aligned} u^{inn} &= & & u^{(1)}(\bar{r},\varphi,s,t) &+\ldots \\ v^{inn} &= \epsilon^{-1} v^{(0)}(\bar{r},s,t) &+& v^{(1)}(\bar{r},\varphi,s,t) &+\ldots \\ w^{inn} &= \epsilon^{-1} w^{(0)}(\bar{r},s,t) &+& w^{(1)}(\bar{r},\varphi,s,t) &+\ldots \end{aligned}$$

  Rem : $u^{(0)} = 0$ assumed
  Leading order axisymmetric
  - Equations + Boundary Conditions at $\bar{r} = 0$
    Equations on $\cos(\varphi)$ and $\sin(\varphi)$ need not BC at $\bar{r} = \infty$!
  - Expansion at at $\bar{r} = \infty$ (Singular integral) + Identification
    $\Rightarrow$ Equation of motion

2. Expansions and Matching

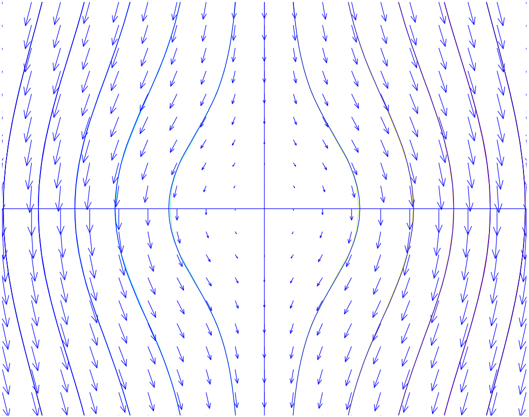


Figure: Relative velocity field : horizontal axis is the normal and vertical axis is the binormal

3. The resulting equation of motion

- Equation of motion for $\mathbf{X}^{(0)}$

$$\dot{\mathbf{X}}^{(0)}(s,t) = \mathbf{Q} + \frac{K(s,t)}{4\pi}\left[-\ln\epsilon + \ln(S) - 1 + C_v(t) + C_w(t)\right]\mathbf{b}(s,t)$$

- Equation of motion for $\mathbf{X}^{(1)}$: Fukumoto and Miyazaki[8] + correcting terms

$$\dot{\mathbf{X}}^{(1)} = \mathbf{E}_1 + c_n\mathbf{n}(s,t) + \left\{C_1^* + \frac{m}{4\pi}KT\left[\ln\epsilon + \frac{5}{6} - \ln S\right]\right\}\mathbf{b}(s,t)$$

$$c_n = C_2^* - \frac{1}{8\pi}K^{(1)}(s,t) + \frac{1}{4\pi}K^{(1)}(s,t)\ln\epsilon - \frac{m}{4\pi}K_s\left[3\ln\epsilon + 3 + \frac{5}{6} - 3\ln S\right]$$

[8] Fukumoto and Miyazaki, "Three dimensional distortions of a vortex filament with axial velocity".

4. Core equations : the compatibility condition

- Compatibility conditions for the symmetrical leading order $v^{(0)}$ and $w^{(0)}$ do not depend on s
  Non-axisymmetric first order only depend on leading order
  Temporal equations for the symmetrical leading order : closed vortex[9]

$$\frac{\partial v^{(0)}}{\partial t} - \alpha^2 \left(\left[\frac{1}{\bar{r}} (\bar{r} v^{(0)})_{\bar{r}} \right]_{\bar{r}} - \frac{v^{(0)}}{\bar{r}^2} \right) = \frac{1}{2} \left(\bar{r} v^{(0)} \right)_{\bar{r}} \frac{\dot{S}^{(0)}}{S^{(0)}}$$

$$\frac{\partial w^{(0)}}{\partial t} - \alpha^2 \frac{1}{\bar{r}} \left[\bar{r} w^{(0)}_{\bar{r}} \right]_{\bar{r}} = \frac{1}{2} \bar{r}^3 \left(\frac{w^{(0)}}{\bar{r}^2} \right)_{\bar{r}} \frac{\dot{S}^{(0)}}{S^{(0)}}$$

$S(t)$ is the length of the closed filament

[9] Callegari and Ting, "Motion of a curved vortex filament with decaying vortical core and axial velocity".

4. Core equations : the compatibility condition

- Compatibility conditions for the axisymmetric part $v_c^{(1)}, w_c^{(1)}$ of first order $v_c^{(1)}$ does not dependent on s and $w_c^{(1)}$ such that :

$$\frac{\partial w_c^{(1)}(\bar{r}, s, t)}{\partial s} = -\dot{\sigma}^{(0)} + \frac{\dot{S}^{(0)}}{S^{(0)}}\sigma^{(0)}$$

Non-axisymmetric second order only depend on leading order
Temporal equations for the axisymmetric part first order : closed vortex[10]

$$\frac{\partial v_c^{(1)}}{\partial t} - \alpha^2 \left(\left[\frac{1}{\bar{r}} (\bar{r} v_c^{(1)})_{\bar{r}} \right]_{\bar{r}} - \frac{v_c^{(1)}}{\bar{r}^2} \right) - \frac{1}{2} \left(\bar{r} v_c^{(1)} \right)_{\bar{r}} \frac{\dot{S}^{(0)}}{S^{(0)}} = c_1^{(1)}(\bar{r}, t)$$

$$\frac{\partial {w_c}^{(1)}}{\partial t} - \alpha^2 \frac{1}{\bar{r}} \left[\bar{r} {w_c}_{\bar{r}}^{(1)} \right]_{\bar{r}} - \frac{1}{2} \bar{r}^3 \left(\frac{{w_c}^{(1)}}{\bar{r}^2} \right)_{\bar{r}} \frac{\dot{S}^{(0)}}{S^{(0)}} = c_2^{(1)}(\bar{r}, t)$$

[10] D. Margerit. "The complete first order expansion of a slender vortex ring". In: *IUTAM Symposium on Dynamics of Slender Vortices*. Ed. by E. Krause and K. Gersten. Aachen: Kluwer academic publishers, 31 Aug.-3 Sep. 1997, pp. 45–54.

Outline

Generalisation, Axial variation

- Short time t/ϵ^2 :
  - Initial velocity of the vortex filament $\neq$ One-time analysis
    Performed in 2-d : Ting and Tung Phy. Fluid[11]
  - leading order not axisymmetric
  - leading order axisymmetric and short waves $\bar{s} = s/\epsilon$: Widnall instability
- Weak Axial variation One-time analysis :
  $\xi = s\epsilon$ Klein and Ting[12]
- Axial variation One-time analysis :
  Klein and Ting, Appl. Math. Lett[13]
  Temporal equations for a leading order compatible that depends on s

[11] L. Ting and C. Tung. "Motion and Decay of a Vortex in a Nonuniform Stream". In: *Motion and Decay of a Vortex in a Nonuniform Stream. The Physics of Fluids* 8.6 (1965), pp. 1039–1051.

[12] L. Ting and R. Klein. *Viscous Vortical Flows (Monograph)*. 181-185. Springer: Lecture Notes in Physics, 1991.

[13] R. Klein and L. Ting. "Vortex filaments with axial core structure variation". In: *Appl.Math.Lett.* 5 (1992), pp. 99–103.

Generalisation, Axial variation

- Ad-hoc studies (toy models) :
  - Lundgren Ashurst : Area-varying waves JFM[14]
  - Marshall : Curved vortices with variable core area JFM[15]
  - Leonard : Nonlocal theory of area-varying waves Phys.Fluids[16]

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[14] T.S. Lundgren and W.T. Ashurts. "Area-varying waves on curved vortex tubes with application to vortex breakdown". In: *Area-varying waves on curved vortex tubes with application to vortex breakdown. J. Fluid Mech.* 200 (1989), pp. 283–307.

[15] J.S. Marshall. "A general theory of curved vortices with circular cross-section and variable core area". In: *A general theory of curved vortices with circular cross-section and variable core area. J. Fluid Mech.* 229 (1991), pp. 311–338.

[16] A. Leonard. "Nonlocal theory of area-varying waves on axisymmetric vortex tubes". In: *Nonlocal theory of area-varying waves on axisymmetric vortex tubes. Phys.Fluids A* 6.2 (1994), pp. 765–777.

Symmetrical axial variations

- Short time $\tau = t/\epsilon$:
  leading order axisymmetric and axial variation
  The compatibility conditions become

$$(\bar{r}u_c^{(1)})_{\bar{r}} + \frac{\bar{r}}{\sigma^{(0)}} w_s^{(0)} = 0$$

$$\frac{\partial v^{(0)}}{\partial \tau} + \frac{\left(\bar{r}v^{(0)}\right)_{\bar{r}}}{\bar{r}} u_c^{(1)} + \frac{w^{(0)}}{\sigma^{(0)}} v_s^{(0)} = 0$$

$$\frac{\partial w^{(0)}}{\partial \tau} + w_{\bar{r}}^{(0)} u_c^{(1)} + \frac{p_s^{(0)}}{\sigma^{(0)}} + \frac{w^{(0)}}{\sigma^{(0)}} w_s^{(0)} = 0$$

$$p^{(0)} = -\int_{\bar{r}}^{\infty} \frac{{v^{(0)}}^2}{\bar{r}} d\bar{r}$$

"Long wave scaling" (Shallow water)

$$u_c^{(1)} = -\frac{1}{\sigma^{(0)}\bar{r}} \psi_s$$

$$w^{(0)} = \frac{1}{\bar{r}} \psi_{\bar{r}}$$

Symmetrical axial variations

$$\begin{aligned} \Gamma &= \bar{r} v^{(0)} \\ y &= \bar{r}^2 \end{aligned}$$

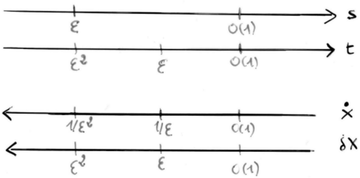
$$\begin{aligned} \frac{\partial \Gamma}{\partial \tau} - \frac{2}{\sigma^{(0)}} \psi_s \Gamma_y + \frac{2}{\sigma^{(0)}} \psi_y \Gamma_s = & \quad 0 \\ D^2 \frac{\partial \psi}{\partial \tau} + \frac{2}{\sigma^{(0)}} \psi_y D^2 \psi_s + \frac{2}{y^2 \sigma^{(0)}} \Gamma \Gamma_s - \frac{2}{\sigma^{(0)}} y \psi_s \left[y^{-1} D^2 \psi \right] = & \quad 0 \\ D^2 \psi = r w_r^{(0)} = 4 y \psi_{yy} & \end{aligned} \tag{1}$$

See Leibovich[17]
length $L = O(1/K) \gg l = O(\delta)$, $l/L = \epsilon$, velocity Γ/L

[17] S. Leibovich. "Weakly non-linear waves in rotating fluids". In: *Weakly non-linear waves in rotating fluids. J. Fluid Mech.* 42 (1970), pp. 803–822.

Symmetrical axial variations

length $L = O(1/K) \gg l = O(\delta)$, $l/L = \epsilon$, velocity Γ/L



Similar vortex

$S(t)$ length of the closed filament

$$v^{(0)} = \frac{1}{2\pi\bar{r}}\left[1 - e^{-\left(\bar{r}/\bar{\delta}\right)^2}\right]$$

$$w^{(0)} = \frac{m_0}{\pi\bar{\delta}^2}\left(\frac{S_0}{S}\right)e^{-\left(\bar{r}/\bar{\delta}\right)^2}$$

$$\bar{\delta}^2 = \frac{S_0}{S}\left(1 + 4\alpha^2\int_0^t \frac{S(t^*)}{S_0}dt^*\right)$$

$$C_v(t) = \left(1 + \gamma - \ln 2\right)/2 - \ln\bar{\delta}$$

$$C_w(t) = -2\left(\frac{S_0}{S}\right)^4\left(m_0/\bar{\delta}\right)^2$$

Similar vortex

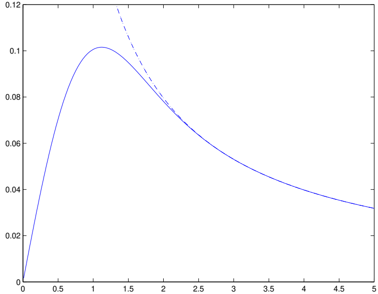


Figure: $v^{(0)}(\bar{r})$

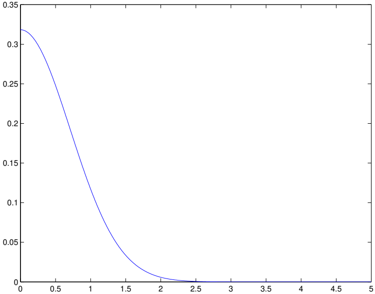


Figure: $w^{(0)}(\bar{r})$

Leading order compatibility conditions with symmetrical axial variation

$$
\begin{aligned}
(\bar{r} u_c^{(1)})_{\bar{r}} + \frac{\bar{r}}{\sigma^{(0)}} w_s^{(0)} &= 0 \\
\frac{\left(\bar{r} v^{(0)}\right)_{\bar{r}}}{\bar{r}} u_c^{(1)} + \frac{w^{(0)}}{\sigma^{(0)}} v_s^{(0)} &= 0 \\
w_{\bar{r}}^{(0)} u_c^{(1)} + \frac{p_s^{(0)}}{\sigma^{(0)}} + \frac{w^{(0)}}{\sigma^{(0)}} w_s^{(0)} &= 0
\end{aligned}
$$