Dear Stephen J. Cowley,

I was very pleased to learn from Oliver Jensen that you would like to invite me in your Tuesday afternoons seminar in the Department of Applied Mathematics and Theoretical Physics. I thank you very much for this. I have already given the date of 1 February to Oliver Jensen.

Here is the Title of my talk :

"Expansion of the Biot and Savart law applied to a curved slender vortex filament with axial variation."

Here is an Abstract :

"We will show how to obtain the two first orders of the expansion of the potential flow induced by a curved vortex line with the distance r to the line as the expansion parameter by using the method of matched asymptotic expansion of singular integral. The order $O(r)$ of this expansion will be given with both its local and global integral parts. This method will also be used to give the inner expansion of the flow induced by a slender vortex with the slenderness $\eps$ as the expansion parameter and an application will be given for a circular vortex ring with axial core structure variation. Finaly we will summarise the derivation of the equation of motion of the central line [Callegari and Ting SIAM J. Appl. Math. 35(1), 1978], give its generalisation at next order and speak about its generalisation to a curved filament with axial variation."

I would like to know if this material corresponds to what you want. I would like to know how many time I have. I would like to know if the audience will essentially composed of persons of your group : High-Reynlds-Number Flow (already familiar with asymptotic descriptions) or if it a much more general audience. Just please let me know.

Thanks you very much.

Yours Sincerely,

Daniel Margerit

Expansion of the Biot-Savart law applied to a curved slender vortex filament with axial variation

D. Margerit
University of Warwick

1. Introduction

2. The potential flow induced by a vortex line

3. The expansion of the velocity field induced by a slender vortex

4. Application to a slender vortex with axial variation

5. Equation of motion of a curved slender filament
   - Equation of Callegari and Ting
   - Cut-off equations

6. Derivation, Assumptions, Next order

7. Generalisation, Axial variation

1 Introduction

- Domaine of study Incompressive laminar flows with vorticity and $Re \gg 1$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\nabla\mathbf{v} = -\nabla\frac{p}{\rho} + \nu\Delta\mathbf{v}$$

$$\nabla \cdot \mathbf{v} = 0$$

- The field of Vorticity

$$\omega = \nabla \times \mathbf{v}$$

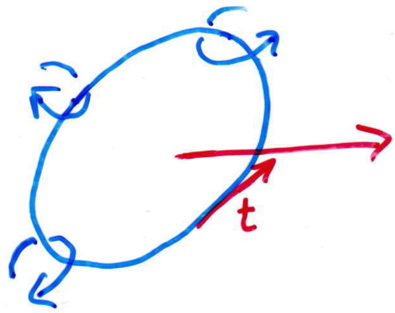
$$\frac{\partial \omega}{\partial t} + \mathbf{v}\nabla\omega = \omega\nabla\mathbf{v} + \nu\Delta\omega$$

$$\mathbf{v}(\mathbf{x}) = \nabla\varphi(\mathbf{x}) + \frac{1}{4\pi}\iiint \frac{\omega(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}',$$

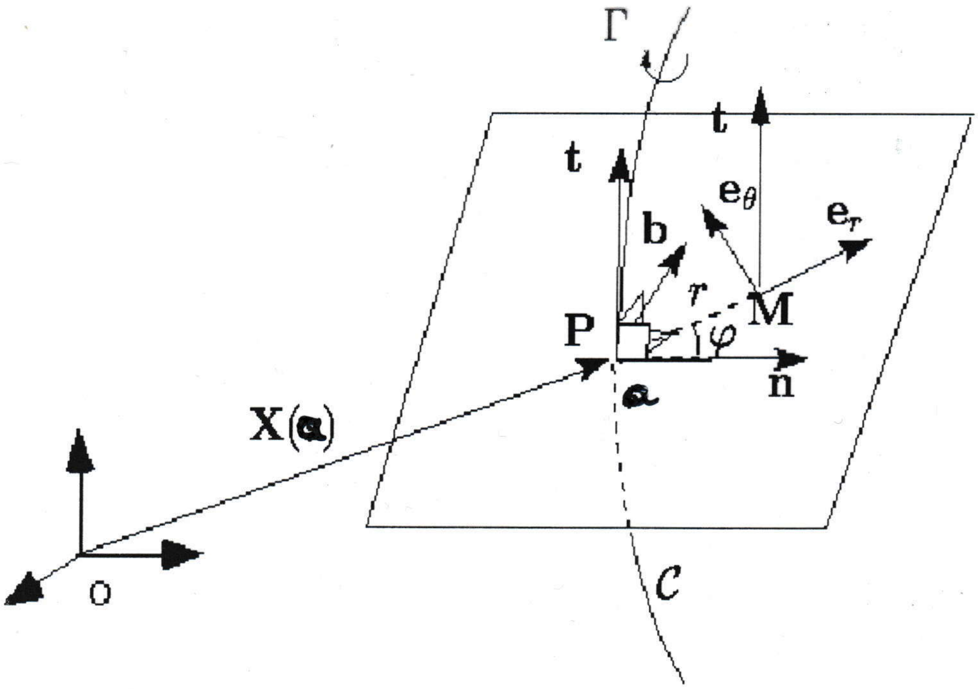
Background velocity $\Delta\varphi = 0$ + Induced velocity

2 The potential flow induced by a vortex line

The curved filament without thickness $\boldsymbol{\omega} = \Gamma \delta_{\mathcal{C}} \mathbf{t}$



$$\mathbf{v}(\mathbf{x}) \quad = \quad \frac{\Gamma}{4\pi} \int_{\mathcal{C}} \frac{\mathbf{t}(a') \times (\mathbf{x} - \mathbf{X}(a'))}{|\mathbf{x} - \mathbf{X}(a')|^3} da'$$



$$\mathbf{v}(r) = \frac{\Gamma}{4\pi r}\mathbf{e}_\theta + \frac{\Gamma}{4\pi} K \ln \frac{\mathcal{L}}{r} \mathbf{b} + \mathbf{Q}_f + O(r)$$

Dimensionless : length $L = O(1/K)$ velocity Γ/L

$$\mathbf{v}(r \to 0, \varphi, a) = \frac{1}{2\pi r}\mathbf{e}_\theta + \frac{K}{4\pi}\cos\varphi\,\mathbf{e}_\theta + \mathbf{A} + \frac{K}{4\pi}\left[\ln\frac{S}{r} - 1\right]\mathbf{b} + r\mathbf{l} + \frac{1}{4\pi}\left(\mathbf{B} - 3\mathbf{C} - \frac{4}{S^2}\mathbf{e}_\theta\right) r + O(r^2 \ln r),$$

where

$$\mathbf{l} = \frac{3}{16}\frac{K^2}{\pi}\left[(\mathbf{e}_r \sin 2\varphi + \mathbf{e}_\theta \cos 2\varphi)\left(\ln\frac{S}{r} - \frac{4}{3}\right)\right] + \frac{3}{16}\frac{K^2}{\pi}\left[\frac{1}{2}\mathbf{e}_\theta \cos 2\varphi + \frac{1}{18}\mathbf{e}_\theta\right] + \frac{1}{4\pi}(K_a \sin\varphi - KT\cos\varphi)\left[\ln\frac{S}{r} - 1\right]\mathbf{t}.$$

Fukumoto and Miyazaki JFM 222, 1991

Result :

$$\mathcal{L} = S$$
$$\mathbf{Q}_f = \frac{\Gamma K}{4\pi}(\cos\varphi\,\mathbf{e}_\theta - \mathbf{b}) + \Gamma\mathbf{A}.$$

Dimensionless : length $L = O(1/K)$ velocity Γ/L

$$\mathbf{v}(r \to 0, \varphi, a) = \boxed{\frac{1}{2\pi r}\mathbf{e}_\theta} + \boxed{\frac{K}{4\pi}\cos\varphi\,\mathbf{e}_\theta + \mathbf{A} + \frac{K}{4\pi}\left[\ln\frac{S}{r} - 1\right]\mathbf{b}} + O(r)$$

where

$$\mathbf{A}(a) = \frac{1}{4\pi}\int_{-S/2}^{+S/2}\left[\frac{\mathbf{t}(a+a^*)\times\mathbf{d}}{|\mathbf{d}|^3} - \frac{K(a)\mathbf{b}(a)}{2|a^*|}\right]da^*,$$

with $\mathbf{d} = \mathbf{X}(a) - \mathbf{X}(a+a^*)$

Result :

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$$\mathbf{A}(a) = \frac{1}{4\pi}\int_{-S/2}^{+S/2}[\frac{\mathbf{t}(a+a^*)\times\mathbf{d}}{|\mathbf{d}|^3} - \frac{K(a)\mathbf{b}(a)}{2|a^*|}]da^*,$$

$$\mathbf{B}(\varphi, a) = \mathbf{e}_r(\varphi, a)\times\int_{-S/2}^{+S/2}[-\frac{\mathbf{t}(a+a^*)}{|\mathbf{d}|^3} - f_b(a, a^*)]da^*,$$

$$\mathbf{C}(\varphi, a) = \int_{-S/2}^{+S/2}[\frac{\mathbf{e}_r(\varphi, a)\cdot\mathbf{d}}{|\mathbf{d}|^5}[\mathbf{t}(a+a^*)\times\mathbf{d}] - f_c(a, a^*)]da^*,$$

with $\mathbf{d} = \mathbf{X}(a) - \mathbf{X}(a+a^*)$ and

$$f_b(a, a^*) = -\frac{1}{|a^*|^3}[\mathbf{t}(a) + K(a)\mathbf{n}(a)a^*], \\ -\frac{1}{|a^*|^3}[\frac{a^{*2}}{2}[K_a(a)\mathbf{n}(a) + K(a)T(a)\mathbf{b}(a) - \frac{3}{4}K^2(a)\mathbf{t}(a)]]$$

$$f_c(a, a^*) = -\frac{K^2(a)\mathbf{b}(a)\cos(\varphi)}{4|a^*|}.$$

MAESI Method

1) Change of variable $a^* = a' - a$ is done :

$$\begin{aligned}\mathbf{v}(r,\varphi,a) &= \frac{1}{4\pi}\int_{\mathcal{C}} \frac{\mathbf{t}(a') \times (\mathbf{x} - \mathbf{X}(a'))}{|\mathbf{x} - \mathbf{X}(a')|^3} da' \\ &= \frac{1}{4\pi}\int_{-S/2}^{+S/2} \mathbf{K}(r,\varphi,a,a^*) da^*,\end{aligned}$$

2) Small intermediate parameter η with $r \ll \eta \ll 1$

$$\mathbf{v}(r,\varphi,a) = \mathbf{Ex} + \mathbf{In}$$

$$\begin{aligned}\mathbf{Ex} &= \frac{1}{4\pi}\int_{-S/2}^{-\eta} \mathbf{K} da^* + \frac{1}{4\pi}\int_{\eta}^{+S/2} \mathbf{K} da^* \\ \mathbf{In} &= \frac{1}{4\pi}\int_{-\eta}^{\eta} \mathbf{K} da^*\end{aligned}$$

3) Stretched variable $\bar{a} = a^*/r$ introduced

$$\mathbf{In} = \frac{1}{4\pi} r \int_{-\eta/r}^{\eta/r} \tilde{\mathbf{K}} d\bar{a}$$

$$\tilde{\mathbf{K}}(r,\varphi,a,\bar{a}) = \mathbf{K}(r,\varphi,a,r\bar{a})$$

4) Outer Part : expansion with r of $\mathbf{Ex}$

- expansion of $\mathbf{K}$ with r and integrate
- expansion with $\eta \ll 1$

Example :

$$\int_{\eta}^{+S/2} \frac{\mathbf{t}(a+a^*) \times \mathbf{d}}{|\mathbf{d}|^3} da^* = \int_{\eta}^{+S/2} \frac{K(a)\mathbf{b}(a)}{2|a^*|} da^* + \int_{\eta}^{+S/2} [\frac{\mathbf{t}(a+a^*) \times \mathbf{d}}{|\mathbf{d}|^3} - \frac{K(a)\mathbf{b}(a)}{2|a^*|}] da^*$$

because

$$\mathbf{t}(a+a^*) \times \mathbf{d}/|\mathbf{d}|^3 = K(a)\mathbf{b}(a)/2|a^*| + O(1)$$

5) Inner part : expansion with r of $\mathbf{In}$

- expansion of $\tilde{\mathbf{K}}$ with r and integrate
- expansion of $\mathbf{In}$ with $\eta/r \gg 1$

6) Addition of $\mathbf{Ex}$ and $\mathbf{In}$: η disappears

3 The expansion of the velocity field induced by a slender vortex

$$\mathbf{v}(\mathbf{x}) = \frac{1}{4\pi}\iiint \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d\mathbf{x}',$$

Dimensionless : length $L = O(1/K)$ velocity Γ/L

- Outer limit : $\epsilon \to 0$ with r fixed: $\mathbf{v}^{out} = \mathbf{v}^{out(0)} + \epsilon \mathbf{v}^{out(1)} + O(\epsilon^2)$
- Inner limit : $\epsilon \to 0$ with $\bar{r} = r/\epsilon$ fixed: $\mathbf{v}^{inn} = \epsilon^{-1}\mathbf{v}^{inn(0)} + \mathbf{v}^{inn(1)} + O(\epsilon)$

$$\mathbf{v}(r,\varphi,s,t,\epsilon) = \frac{1}{4\pi}\iiint \frac{\epsilon^2 \boldsymbol{\omega}(\bar{r}',\varphi',a',\epsilon) \times \left[\mathbf{X} + r\mathbf{e}_r - (\mathbf{X}' + \epsilon\bar{r}'\mathbf{e}_r')\right]}{\left|\mathbf{X} + r\mathbf{e}_r - (\mathbf{X}' + \epsilon\bar{r}'\mathbf{e}_r')\right|^3} h_3'\bar{r}'d\bar{r}'d\varphi'da',$$

where $h_3' = (1 - K(a')\epsilon\bar{r}'\cos(\varphi'))$.

Problem : Velocity induced by $\boldsymbol{\omega}(\mathbf{x}) = \boldsymbol{\omega}(\bar{r},\varphi,a,\epsilon) = \epsilon^{-2}\boldsymbol{\omega}^{(0)}(\bar{r},\varphi,a)$

- $\mathbf{v}^{out(0)}$: line vortex, $\mathbf{v}^{out(1)} \neq 0$

- MAESI method in ϵ :

$$\mathbf{v}^{inn(0)} = -\frac{1}{2\pi}\iint \mathbf{g}\bar{r}'d\bar{r}'d\varphi'$$

$$\mathbf{v}^{inn(1)} =$$

$$\boxed{\mathbf{A} + \frac{K}{4\pi}[\ln\frac{S}{\epsilon} - 1]\mathbf{b}}$$
$$-\frac{1}{4\pi}\iint \boldsymbol{\omega}_a^{(0)}(\bar{r}',\varphi',a)\times\mathbf{t}(a)\ln\frac{1}{k^2}\bar{r}'d\bar{r}'d\varphi'$$
$$-\frac{K(a)}{8\pi}\iint \boldsymbol{\omega}^{(0)}(\bar{r}',\varphi',a)\times\mathbf{n}(a)\ln\frac{1}{k^2}\bar{r}'d\bar{r}'d\varphi'$$
$$-\frac{K(a)\bar{r}\cos(\varphi)}{4\pi}\iint \mathbf{g}\bar{r}'d\bar{r}'d\varphi'$$
$$-\frac{T(a)\bar{r}}{2\pi}\iint \frac{\boldsymbol{\omega}^{(0)}(\bar{r}',\varphi',a)\times\mathbf{t}(a)}{k^2}\sin(\varphi-\varphi')\bar{r}'^2d\bar{r}'d\varphi'$$
$$+\frac{K(a)}{4\pi}\iint \mathbf{g}\cos\varphi'\bar{r}'^2d\bar{r}'d\varphi',$$

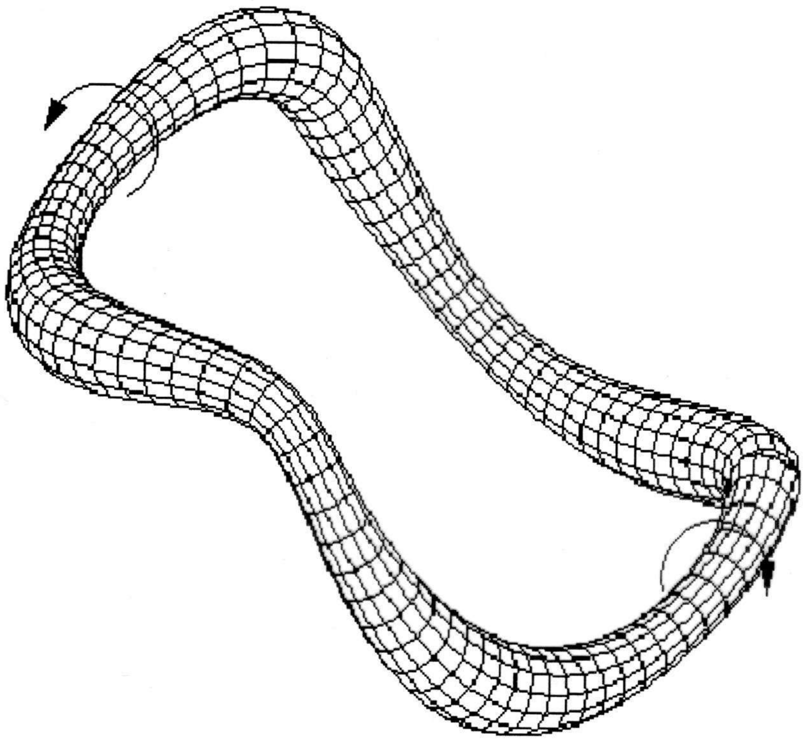
where

$$\mathbf{g} = \boldsymbol{\omega}^{(0)}(\bar{r}',\varphi',a)\times\left[\bar{r}'\mathbf{e}_r(\varphi',a) - \bar{r}\mathbf{e}_r(\varphi,a)\right]/k^2$$
$$k^2 = \bar{r}^2 + \bar{r}'^2 - 2\bar{r}\bar{r}'\cos(\varphi-\varphi')$$

4 Application to a slender vortex with axial variation

- Chosen vorticity

$$\boldsymbol{\omega} = \frac{1}{\epsilon^2}\left[\frac{1}{\pi\bar{r}_0^2(a)}\mathbf{t} + \epsilon\frac{\bar{r}_0'(a)}{\pi\bar{r}_0^3(a)}\frac{\bar{r}}{h_3}\mathbf{e}_r + \frac{g(a,\bar{r})}{h_3}\mathbf{e}_\theta\right] H\left(1 - \frac{\bar{r}}{\bar{r}_0(a)}\right)$$



div$\boldsymbol{\omega} = 0$ and $\boldsymbol{\omega}\cdot\mathbf{N} = 0$ on $\bar{r} = \bar{r}_0(a)$

- Decomposition $\mathbf{v} = \mathbf{v}(\bar{r} = 0, a) + \mathbf{V}$

- Induced velocity : $(g(a,\bar{r})=0)$

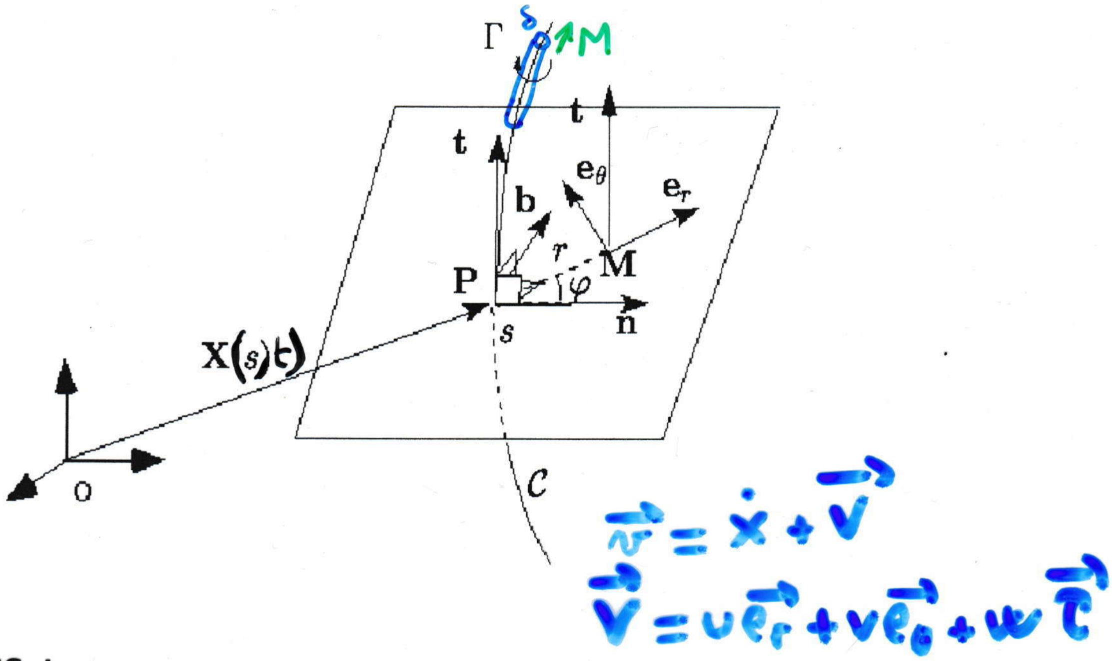
$$\mathbf{V}^{inn(0)} = \frac{\bar{r}}{2\pi\bar{r}_0^2}\mathbf{e}_\theta \text{ if } \bar{r} < \bar{r}_0(s)$$
$$= \frac{1}{2\pi\bar{r}}\mathbf{e}_\theta \text{ if } \bar{r} > \bar{r}_0(s),$$

$$\mathbf{V}^{inn(1)} = -\left(\frac{\bar{r}}{\bar{r}_0}\right)^2 \frac{K}{16\pi}\left[3\sin\varphi\mathbf{e}_r + \cos\varphi\mathbf{e}_\theta\right] \qquad \text{if } \bar{r} < \bar{r}_0(s)$$
$$= -\frac{K}{16\pi}\left(\left[-\left(\frac{\bar{r}_0}{\bar{r}}\right)^2 + 4 + 4\ln\frac{\bar{r}}{\bar{r}_0}\right]\sin\varphi\mathbf{e}_r + \left[\left(\frac{\bar{r}_0}{\bar{r}}\right)^2 + 4\ln\frac{\bar{r}}{\bar{r}_0}\right]\cos\varphi\mathbf{e}_\theta\right) \text{ if } \bar{r} > \bar{r}_0(s)$$

and

$$\mathbf{v}(\bar{r}=0,a) = \mathbf{A} + \frac{K\mathbf{b}}{4\pi}\left[\ln\frac{S}{(\epsilon\bar{r}_0)}\right].$$

5 Equation of motion of a curved slender filament



Dimensionless :

length $L = O(1/K) \gg l = O(\delta)$, $l/L = \epsilon$, velocity Γ/L

$$Re = \frac{\Gamma}{\nu} = \frac{1}{\alpha^2\epsilon^2} \gg 1 \qquad Sw = \frac{\Gamma\delta}{M} = \frac{\epsilon}{M} = \frac{1}{m} + O(\epsilon)$$

- Equation of Callegari and Ting

$$\dot{\mathbf{X}}(s,t) = \mathbf{Q} + \frac{K(s,t)}{4\pi}\left[\ln(S/\epsilon) - 1 + C_v(t) + C_w(t)\right]\mathbf{b}(s,t)$$

$$\text{with } \mathbf{Q} = \mathbf{A}(s,t) - [\mathbf{A}(s,t)\cdot\mathbf{t}(s,t)]\,\mathbf{t}(s,t)$$

$$\mathbf{A}(s,t) = \frac{1}{4\pi}\int_{-\pi}^{+\pi} ds'\ \sigma' \left[\frac{\mathbf{t}(s+s',t)\times\mathbf{d}}{|\mathbf{d}|^3} - \frac{K(s,t)\mathbf{b}(s,t)}{2\,|\lambda(s,s',t)|}\right]$$

$$\mathbf{d} = \mathbf{X}(s,t) - \mathbf{X}(s+s',t)$$

- Cut-off equations

$$\partial \mathbf{X}/\partial t = \frac{1}{4\pi} \int_I \sigma(s',t) \frac{\mathbf{t}(s',t) \times (\mathbf{X}(s,t) - \mathbf{X}(s',t))}{|\mathbf{X}(s,t) - \mathbf{X}(s',t)|^3} ds'$$

where $I = [0, 2\pi[\backslash [s - s_c, s + s_c[$ $\quad s_c$: cut-off length

Singular Integral for the small parameter s_c

MAESI + Comparison

$$s_c(s,t) = \epsilon \exp\left(1 - \ln 2 - C_v(t) - C_w(t)\right) / \sigma(s,t)$$

Regularization methods

6 Derivation, Assumptions, Next order

- Equations on curvilinear coordinates (r, φ, s)

$$\begin{aligned} \mathbf{v}(r,\varphi,s,t,\epsilon) &= \dot{\mathbf{X}}(s,t,\epsilon) + \mathbf{V}(r,\varphi,s,t,\epsilon) \\ \mathbf{V} &= u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{t} \end{aligned}$$

  - Biot Savart
  - Continuity

$$(urh_3)_r + (h_3 v)_\varphi + rw_s - Trw_\varphi = -r\dot{\mathbf{X}}_s \cdot \mathbf{t}$$

where $h_3 = \sigma\left(1 - rK(s)\cos(\varphi)\right)$ and $\sigma = |\mathbf{X}_s|$

  - Navier-Stokes

$$\begin{aligned} \mathbf{a} &= -\nabla p + \nu\Delta\mathbf{V} + \frac{\nu}{h_3}\left(\frac{1}{h_3}\dot{\mathbf{X}}_s\right)_s \\ \mathbf{a} &= \left(\frac{\partial \mathbf{V}}{\partial t}\right)_{r,\varphi,s} + (\mathbf{V} - r\dot{\mathbf{e}}_r)\cdot\nabla\mathbf{V} + \frac{\dot{\mathbf{X}}_s}{h_3}(w - r\dot{\mathbf{e}}_r \cdot \mathbf{t}) \end{aligned}$$

$$\left(\frac{\partial \mathbf{V}}{\partial t}\right)_{r,\varphi,s} = \frac{\partial u}{\partial t}\mathbf{e}_r + u\dot{\mathbf{e}}_r + \ldots$$

- Expansions and Matching

$$\mathbf{X} = \mathbf{X}^{(0)}(s,t) + \epsilon\mathbf{X}^{(1)}(s,t) + \ldots$$

– Outer limit: $\epsilon \to 0$ with r fixed:

$$\mathbf{v}^{out} = \mathbf{v}^{out(0)} + \epsilon\mathbf{v}^{out(1)} + O(\epsilon^2)$$

◇ Expansion in the Biot Savart law

◇ Matching law:

expansion at $r = 0$ and replace $r = \bar{r}\epsilon$

$\Rightarrow$ Boundary Conditions at $\bar{r} = \infty$

Only $\cos(\varphi)$ and $\sin(\varphi)$ parts depend on the unknown **X** !

– Inner limit: $\epsilon \to 0$ with $\bar{r} = r/\epsilon$ fixed:

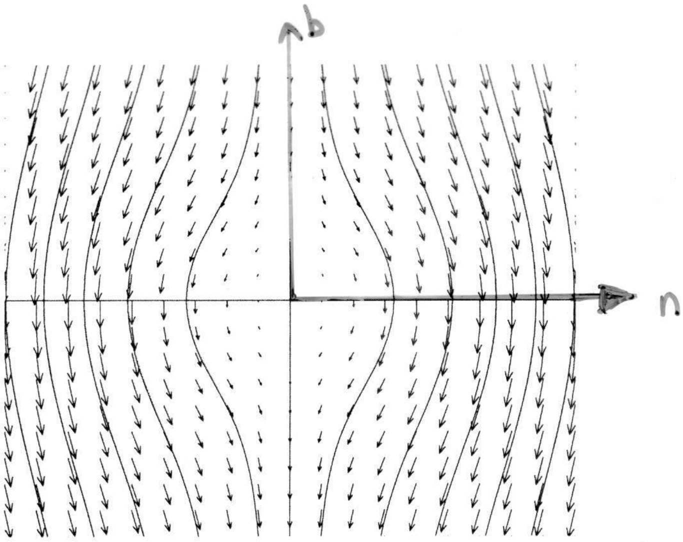
$$\begin{array}{lclclc} u^{inn} & = & & & u^{(1)}(\bar{r},\varphi,s,t) & +\ldots \\ v^{inn} & = & \epsilon^{-1}v^{(0)}(\bar{r},s,t) & + & v^{(1)}(\bar{r},\varphi,s,t) & +\ldots \\ w^{inn} & = & \epsilon^{-1}w^{(0)}(\bar{r},s,t) & + & w^{(1)}(\bar{r},\varphi,s,t) & +\ldots \end{array}$$

Rem : $u^{(0)} = 0$ assumed

Leading order axisymmetric

◇ Equations + Boundary Conditions at $\bar{r} = 0$

Equations on $\cos(\varphi)$ and $\sin(\varphi)$ need not BC at $\bar{r} = \infty$!



◇ Expansion at at $\bar{r} = \infty$ (Singular integral) + Identification

$\Rightarrow$ Equation of motion

Equation of motion for $\mathbf{X}^{(0)}$

$$\dot{\mathbf{X}}(s,t) = \mathbf{Q} + \frac{K(s,t)}{4\pi}\left[-\ln\epsilon + \ln(S) - 1 + C_v(t) + C_w(t)\right]\mathbf{b}(s,t)$$

Equation of motion for $\mathbf{X}^{(1)}$:Fukumoto and Miyazaki + correcting terms

$$\dot{\mathbf{X}}^{(1)} = \mathbf{E}_1 + c_n\mathbf{n}(s,t) + \left\{C_1^* + \frac{m}{4\pi}KT\left[\ln\epsilon + \frac{5}{6} - \ln S\right]\right\}\mathbf{b}(s,t)$$

$$c_n = C_2^* - \frac{1}{8\pi}K^{(1)}(s,t) + \frac{1}{4\pi}K^{(1)}(s,t)\ln\epsilon - \frac{m}{4\pi}K_s\left[3\ln\epsilon + 3 + \frac{5}{6} - 3\ln S\right]$$

Compatibility conditions for the symmetrical leading order :

$v^{(0)}$ and $w^{(0)}$ do not depend on s

Non-axisymmetric first order only depend on leading order

Temporal equations for the symmetrical leading order : closed vortex

$$\frac{\partial v^{(0)}}{\partial t} - \alpha^2 \left(\left[\frac{1}{\bar{r}} (\bar{r} v^{(0)})_{\bar{r}} \right]_{\bar{r}} - \frac{v^{(0)}}{\bar{r}^2} \right) = \frac{1}{2} \left(\bar{r} v^{(0)} \right)_{\bar{r}} \frac{\dot{S}^{(0)}}{S^{(0)}}$$

$$\frac{\partial w^{(0)}}{\partial t} - \alpha^2 \frac{1}{\bar{r}} \left[\bar{r} w^{(0)}_{\bar{r}} \right]_{\bar{r}} = \frac{1}{2} \bar{r}^3 \left(\frac{w^{(0)}}{\bar{r}^2} \right)_{\bar{r}} \frac{\dot{S}^{(0)}}{S^{(0)}}$$

Compatibility conditions for the axisymmetric part $v_c^{(1)}, w_c^{(1)}$ of first order :

$v_c^{(1)}$ does not dependent on s and $w_c^{(1)}$ such that :

$$\frac{\partial w_c^{(1)}(\bar{r}, s, t)}{\partial s} = -\dot{\sigma}^{(0)} + \frac{\dot{S}^{(0)}}{S^{(0)}}\sigma^{(0)}$$

Non-axisymmetric second order only depend on leading order

Temporal equations for the axisymmetric part first order : closed vortex

$$\frac{\partial v_c^{(1)}}{\partial t} - \alpha^2 \left(\left[\frac{1}{\bar{r}} (\bar{r} v_c^{(1)})_{\bar{r}} \right]_{\bar{r}} - \frac{v_c^{(1)}}{\bar{r}^2} \right) - \frac{1}{2} \left(\bar{r} v_c^{(1)} \right)_{\bar{r}} \frac{\dot{S}^{(0)}}{S^{(0)}} = c_1^{(1)}(\bar{r}, t)$$

$$\frac{\partial {w_c}^{(1)}}{\partial t} - \alpha^2 \frac{1}{\bar{r}} \left[\bar{r} {w_c}_{\bar{r}}^{(1)} \right]_{\bar{r}} - \frac{1}{2} \bar{r}^3 \left(\frac{{w_c}^{(1)}}{\bar{r}^2} \right)_{\bar{r}} \frac{\dot{S}^{(0)}}{S^{(0)}} = c_2^{(1)}(\bar{r}, t)$$

7 Generalisation, Axial variation

- Short time t/ϵ^2 :
  - Initial velocity of the vortex filament $\neq$ One-time analysis
    Performed in 2-d : Ting and Tung Phy. Fluid 65
  - leading order not axisymmetric
  - leading order axisymmetric and short waves $\bar{s} = s/\epsilon$: Widnall instability

- Weak Axial variation One-time analysis :

  $\xi = s\epsilon$ Klein and Ting 91

- Axial variation One-time analysis :

  Klein and Ting, Appl. Math. Lett. 92

  Temporal equations for a leading order compatible that depends on s

- Ad-hoc studies (toy models) :
  - Lundgren Ashurst : Area-varying waves JFM 89
  - Marshall : Curved vortices with variable core area JFM 91
  - Leonard : Nonlocal theory of area-varying waves Phys.Fluids 94

- Short time $\tau = t/\epsilon$:

  leading order axisymmetric and axial variation

  The compatibility conditions become

$$
\begin{aligned}
(\bar{r}u_c^{(1)})_{\bar{r}} + \frac{\bar{r}}{\sigma^{(0)}} w_s^{(0)} &= 0 \\
\frac{\partial v^{(0)}}{\partial \tau} + \frac{\left(\bar{r}v^{(0)}\right)_{\bar{r}}}{\bar{r}} u_c^{(1)} + \frac{w^{(0)}}{\sigma^{(0)}} v_s^{(0)} &= 0 \\
\frac{\partial w^{(0)}}{\partial \tau} + w_{\bar{r}}^{(0)} u_c^{(1)} + \frac{p_s^{(0)}}{\sigma^{(0)}} + \frac{w^{(0)}}{\sigma^{(0)}} w_s^{(0)} &= 0
\end{aligned}
$$

$$
p^{(0)} = -\int_{\bar{r}}^{\infty} \frac{{v^{(0)}}^2}{\bar{r}} d\bar{r}
$$

"Long wave scaling" (Shallow water)

$$
\begin{aligned}
u_c^{(1)} &= -\frac{1}{\sigma^{(0)}\bar{r}} \psi_s \\
w^{(0)} &= \frac{1}{\bar{r}} \psi_{\bar{r}}
\end{aligned}
$$

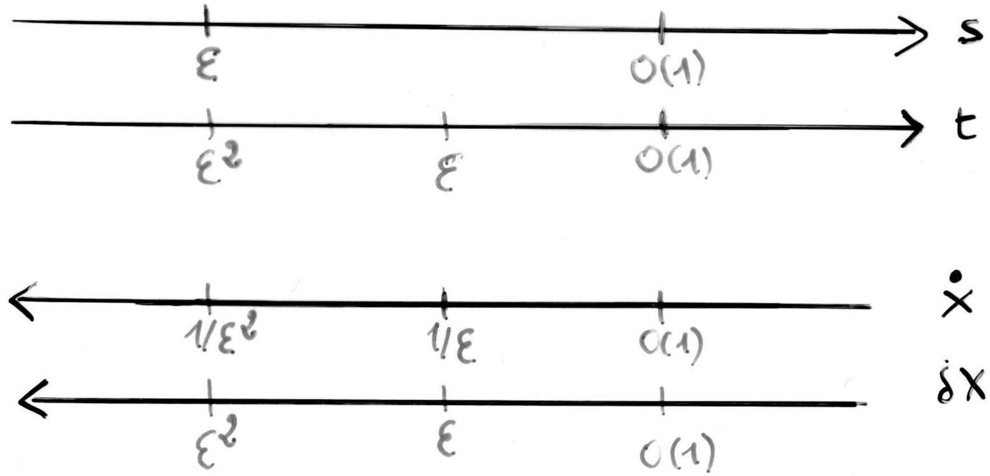
$$
\begin{aligned}
\Gamma &= \bar{r}v^{(0)} \\
y &= \bar{r}^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \Gamma}{\partial \tau} - \frac{2}{\sigma^{(0)}}\psi_s \Gamma_y + \frac{2}{\sigma^{(0)}}\psi_y \Gamma_s &= 0 \\
D^2 \frac{\partial \psi}{\partial \tau} + \frac{2}{\sigma^{(0)}}\psi_y D^2 \psi_s + \frac{2}{y^2 \sigma^{(0)}}\Gamma \Gamma_s - \frac{2}{\sigma^{(0)}} y \psi_s \left[y^{-1} D^2 \psi \right] &= 0
\end{aligned}
$$

$$D^2 \psi = r w_r^{(0)} = 4 y \psi_{yy}$$

Leibovich JFM 70

length $L = O(1/K) \gg l = O(\delta)$, $l/L = \epsilon$, velocity Γ/L



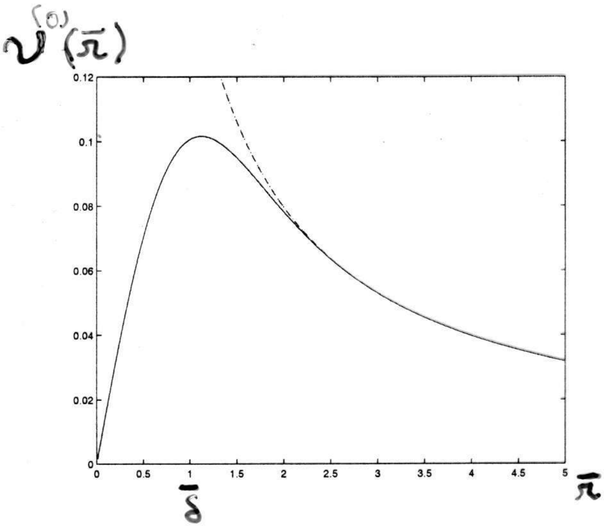
- Similar vortex:

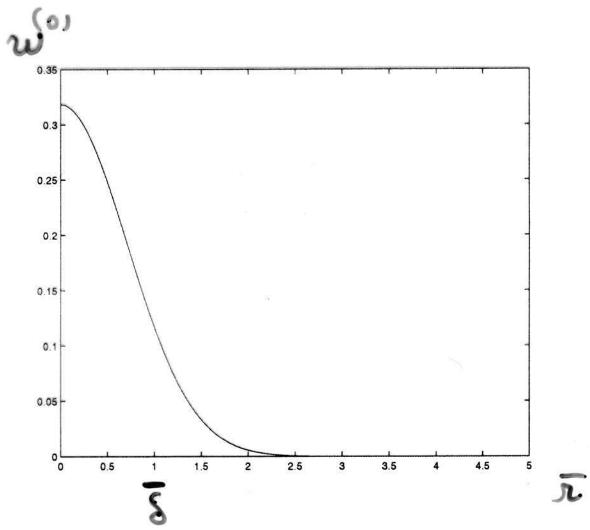
$S(t)$ length of the closed filament

$$v^{(0)} = \frac{1}{2\pi\bar{r}}\left[1 - e^{-\left(\bar{r}/\bar{\delta}\right)^2}\right]$$

$$w^{(0)} = \frac{m_0}{\pi\bar{\delta}^2}\left(\frac{S_0}{S}\right)e^{-\left(\bar{r}/\bar{\delta}\right)^2}$$

$$\bar{\delta}^2 = \frac{S_0}{S}\left(1 + 4\alpha^2\int_0^t \frac{S(t^*)}{S_0}dt^*\right)$$





$$C_v(t) = \left(1 + \gamma - \ln 2\right)/2 - \ln\bar{\delta}$$

$$C_w(t) = -2\left(\frac{S_0}{S}\right)^4\left(m_0/\bar{\delta}\right)^2$$

- Equations of Compatibility :

$$
\begin{aligned}
(\bar{r}u_c^{(1)})_{\bar{r}} + \frac{\bar{r}}{\sigma^{(0)}} w_s^{(0)} &= 0 \\
\frac{\left(\bar{r}v^{(0)}\right)_{\bar{r}}}{\bar{r}} u_c^{(1)} + \frac{w^{(0)}}{\sigma^{(0)}} v_s^{(0)} &= 0 \\
w_{\bar{r}}^{(0)} u_c^{(1)} + \frac{p_s^{(0)}}{\sigma^{(0)}} + \frac{w^{(0)}}{\sigma^{(0)}} w_s^{(0)} &= 0
\end{aligned}
$$

where $\mathbf{v} = \dot{\mathbf{x}} + \mathbf{V}$

$\mathbf{V} = u\, \mathbf{e}_r + v\, \mathbf{e}_\theta + w\, \mathbf{t}$