Axial core-variations of axisymmetric shape on a curved slender vortex filament with a similar, Rankine, or bubble core

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(Received 31 July 2001; accepted 28 August 2002; published 7 November 2002)

The dynamics of axial core-variations of axisymmetric shape on a vortex filament is derived from the Navier–Stokes equations in the slenderness limit. The core of the vortex is of similar, Rankine, or bubble type with a centerline of any shape. In this limit, a two-time-scale asymptotic approach is used to study the dynamics of the axial core-variations and of the centerline. The short-time dynamical equations of the axial core-variations are given and are inviscid at leading and first orders. The induced short-time and normal-time dynamics of the centerline is obtained. The full two-time-scale dynamics of the axial variations and of the filament motion is discussed qualitatively. The normal-time dynamics of vortex filaments without axial core-variations is given in a short form. Within the two-time-scale framework, the dynamics of axial core-variations around this one-time base flow is then studied in the small amplitude limit. The normal-time equations of a vortex bubble are given. The bubble has no axial variations, a centerline of any shape and can have a nonpotential core. The equation for the ultra-short-time dynamics of axial variations on this bubble is given. [DOI: 10.1063/1.1516210]

I. INTRODUCTION

Slender vortex filaments are high concentration of vorticity along a geometrical curve of the fluid flow; they have been studied since more than a century, as their dynamics gives the one of the flow. Almost all the vorticity is inside a tube of small thickness δ compared to the characteristic radius of curvature R of its centerline; the ratio δ/R defines a small parameter ε. In the small thickness limit, this tube is a ''boundary layer'' (in the singular perturbation method point of view); and, except for a straight filament, it is moving with a velocity that depends on its core structure, as can be found from the Biot–Savart law of induction.[1,2] This small moving region of the flow is difficult to track in experimental measurements and induces stiffness in direct numerical simulations of vortex filaments.

Asymptotic methods have been used to get ride of this stiffness[1,2] and to extract, at leading order, a nonstiff equation for the filament motion. This asymptotic theory assumes that the leading-order core is axisymmetric and without axial core-variations. Generalized systems of equations for filaments with axial core-variations have been proposed.[3–6] In these studies, the characteristic length of the axial variations is of the order of the radius of curvature which is bigger than the thickness δ; short wavelengths are not taken into account. These systems are ad hoc equations because they are not asymptotically derived from the Navier–Stokes (N–S) equations.

For a *straight* vortex filament with axial core-variations (bulging waves) Melander and Hussain[7] give a spectral computation of the axisymmetric equations. They compute a bulging wave of wavelength $\approx 11\delta$ and amplitude $\approx 0.5\delta$ at $\mathrm{Re}=\Gamma/\nu\approx 665$, where δ is the mean core radius, Γ is the circulation of the vortex filament, and ν is the kinematic viscosity. In dimensionless variables with the wavelength $L=11\delta$ as the characteristic length, these parameters of simulation correspond to $\varepsilon=\delta/L=1/11\approx 0.09$, and to a viscous parameter (see Sec. II) $\alpha\equiv \mathrm{Re}^{-1/2}/\varepsilon\approx 0.43$. Arendt *et al.*[8] give the evolution of any initial axial variations of small amplitude in the dynamics of the linearized axisymmetric equations.

In this paper I derive and study the leading-order set of equations of motion of *curved* slender vortex filaments of *any shape* with *axial core-variations*. From the study of the eigen-oscillations of a *circular* vortex ring, Kopiev and Chernyshev[9] give the dynamics of the bending modes that oscillates on the time t of the motion of the vortex ring. The dynamics of these bending modes can also be found from a linear stability study of the asymptotic equation of motion of vortex filaments.[10,11] For a circular vortex ring Kopiev and Chernyshev[9] also give the dynamics of bulging modes that can be shown to oscillate on the short time $\tau=t/\varepsilon$. It shows that a two-time-scale analysis can be used to study them for a general curved filament. This would extend the one-time analysis of the previous asymptotic theory[1] without axial core-variations. It also extends to a curved filament the one- and short-time analysis by Souza[12] of a straight filament with core-variations. In this paper, I carry out this two-time-scale analysis.

The paper is organized as follows. In Sec. II, I give the geometrical description of the centerline and of the vortex core in local coordinates near the centerline of the filament. In these coordinates, I give the dynamical equations of the velocity field, those of the interfaces (if there are any: i.e., vortex sheet when there is a jump of axial velocity or bubble free-boundary), and all the two-time-scale asymptotic expansions. In Sec. III the cascade of asymptotic equations is given in the two-time-scale framework. From the axisymmetric part of the first-order asymptotic equations, I obtain short-time-scale dynamical equations for the axial core-variations of axisymmetric shape. These equations are the same as those obtained by Souza[12] for a straight filament. For a curved filament they were first given in Margerit and Brancher.[13] Souza pointed out (private communication) that the equations for a straight filament might be relevant to a curved filament. Here, I prove that curvature does not give adding terms in these equations and I also give the induced short-time dynamics of the curved vortex centerline. In Sec. IV the one-time (normal-time) dynamics of vortex filaments without axial core-variations of the Callegari and Ting theory[1] is given in a short form. This gives the one-time base flow that is used in the linear stability study of Sec. V. This analysis is carried out in the two-time-scale framework to study the dynamics of small axial core-variations around this one-time base flow. In Sec. VI the one-time (normal-time) continuous vortex-core of the Callegari and Ting theory[1] is extended for a vortex bubble with a centerline of any shape and with a nonpotential core. An equation for the ultra-fast dynamics of axial variations of the bubble free-boundary is given. This generalizes the theory of Genoux[14–16] of vortex bubbles of circular centerline and potential core. Finally a conclusion is given in Sec. VII.

Several steps of the derivations are given in the appendices. Appendix A gives the asymmetric part of the equations at first order and Appendix B gives the axisymmetric part of the equations at second order. In Appendix C the core without axial variations appears to be the unique stationary solution of the short-time-scale equations given in Sec. III. Finally in Appendix D the axisymmetric part of the stationary solution of the short-time-scale equations at next order is proven to be the sum of a part without axial variations and of a part with axial variations. The structure of this second part is unique and is induced from the local stretching of the centerline. Fortunately this structure was that introduced in Margerit[17] to generalize the Callegari and Ting theory[1] at next order.

II. NOTATIONS AND TWO-TIME EXPANSIONS

Here, I give the geometrical description of the flow field and of the filament, and the local coordinates that are used. A discussion of the characteristic scales of the asymptotic slender filament regime and the basic assumptions of the asymptotic study are then given. Finally, the equations for the flow are written on the local coordinates and the two-time-scale expansions are given.

The closed centerline of the slender vortex of circulation Γ and length S is described by the vector function $\mathbf{X}=\mathbf{X}(s,t)$ where s stands for the arc-length at $t=0$. At each point of this curve the Frenet vector basis ($\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$) exists with, respectively, the tangent, normal, and binormal vectors. I introduce a *local curvilinear coordinate system* $\mathbf{M}(r,\varphi,s)$ and the curvilinear vector basis $(\mathbf{e}_r,\mathbf{e}_\varphi,\mathbf{t})$ valid near this line. This system is defined in the following manner; if $\mathbf{P}(s)$ is the projection on the centerline of a point $\mathbf{M}$ then $\mathbf{PM}$ is in the plane ($\mathbf{n}$, $\mathbf{b}$), and thus polar coordinates (r, φ) can be used in this plane with the associated polar vectors $(\mathbf{e}_r,\mathbf{e}_\varphi)$ and with φ the angle between $\mathbf{n}$ and $\mathbf{PM}$.

The relative velocity $\mathbf{V}$ is defined by

$$\mathbf{v}=\dot{\mathbf{X}}(s,t)+\mathbf{V}(r,\varphi,s,t), \tag{1}$$

where $\mathbf{v}$ is the velocity of the fluid and $\dot{\mathbf{X}}$ the local filament velocity. The radial, circumferential, and axial components (u,v,w) of the relative velocity are defined by $\mathbf{V}=u\mathbf{e}_r+v\mathbf{e}_\varphi+w\mathbf{t}$. The vorticity field $\boldsymbol{\omega}$ is given by $\boldsymbol{\omega}=\boldsymbol{\nabla}\times\mathbf{V}+\mathbf{t}\times\dot{\mathbf{X}}_s/h_3$, where $h_3=\sigma(1-rK(s)\cos(\varphi))$, K is the local curvature, and $\sigma=|\mathbf{X}_s|$.

In the asymptotic theory of vortex motion, the thickness δ of the ring is of order l and the other length scales, for example the local radius of curvature $1/K$ or the length S of the closed filament are of the same order L. Since the vortex is slender the small parameter $\varepsilon\ll 1$ is defined as the ratio l/L. I nondimensionalize the velocity field with Γ/L, all lengths with L, and the time with L^2/Γ. The *outer problem* is defined by the *outer limit*: $\varepsilon\to 0$ with r fixed, which describes the flow far from the centerline; and the *inner problem* by the *inner limit*: $\varepsilon\to 0$ with $\overline{r}=r/\varepsilon$ fixed, which describes the flow near the centerline.

The Reynolds number $R_e=\Gamma/\nu$, where ν is the kinematic viscosity, is related to ε by $R_e^{-1/2}=\alpha\varepsilon$. Here, the *viscous number* $\alpha=O(1)$ is defined by $\alpha^2=\overline{\nu}/\Gamma$, where $\overline{\nu}=\nu/\varepsilon^2$. The inviscid vortex ring is obtained in the limit $\alpha=0$. The asymptotic ansatz based on the small slenderness ratio allows to unify the related analyses for the Navier–Stokes and Euler equations.

In this study I assume that the vorticity field is centered near the centerline and rapidly decays at large distance. I will assume that the vorticity distribution is of bounded support or decays exponentially. The same standard assumption is also taken for the axial velocity field.

A. General equations

The continuity equation in these curvilinear coordinates (r,φ,s) is[1]

$$(urh_3)_r+(h_3v)_\varphi+rw_s-Tr\,w_\varphi=-r\,\dot{\mathbf{X}}_s\cdot\mathbf{t}, \tag{2}$$

where T is the local torsion of the filament. The Navier–Stokes equation becomes[1]

$$\mathbf{a}=-\nabla p+\nu\Delta\mathbf{V}+\frac{\nu}{h_3}\left(\frac{1}{h_3}\dot{\mathbf{X}}_s\right)_s, \tag{3}$$

where ν is the kinematic viscosity, and the acceleration $\mathbf{a}$ is

$$\mathbf{a}=\left(\frac{\partial\mathbf{V}}{\partial t}\right)_{r,\varphi,s}+(\mathbf{V}-r\dot{\mathbf{e}}_r)\cdot\nabla\mathbf{V}+\frac{\dot{\mathbf{X}}s}{h_3}(w-r\dot{\mathbf{e}}_r\cdot\mathbf{t})+\ddot{\mathbf{X}},$$

with

$$\left(\frac{\partial\mathbf{V}}{\partial t}\right)_{r,\varphi,s}=\frac{\partial u}{\partial t}\mathbf{e}_r+u\dot{\mathbf{e}}_r+\dots.$$

Here and in the whole paper, the pressure is in fact the pressure divided by the constant density of the incompressible fluid. The boundary conditions of these equations are $u=v=0$ at $r=0$.

B. Discontinuous vorticity field

In an *inviscid* fluid, if the vorticity is inside a tube of thickness δ^t, the location of this interface $r=\delta^t(\varphi,s,t)$ is an unknown function. At this interface one has to satisfy both the continuity of pressure

$$[[p]]\equiv p(\delta^{t+})-p(\delta^{t-})=0, \tag{4}$$

and the continuity of the normal velocity

$$[[\mathbf{v}\cdot\mathbf{N}]]=0, \tag{5}$$

where $\mathbf{N}$ is the normal vector to the interface and is given by $\mathbf{N}=-\nabla F/|\nabla F|$ with $F=\delta^t(\varphi,s,t)-r$ and

$$\nabla F=\left(-\mathbf{e}_r+\frac{1}{\delta^t}\frac{\partial\delta^t}{\partial\varphi}\mathbf{e}_\varphi+\frac{1}{h_3}\left(\frac{\partial\delta^t}{\partial s}-\sigma T\frac{\partial\delta^t}{\partial\varphi}\right)\mathbf{t}\right).$$

The dynamical equation for the interface is given by the kinematic boundary condition

$$\frac{\partial\delta^t}{\partial t}+(\mathbf{V}(\delta^{t\pm})-\delta^t\dot{\mathbf{e}}_r)\cdot\nabla F=0, \tag{6}$$

where $\delta^{t\pm}$ is used to allow the possibility of the interface to be a vortex sheet of strength $\gamma=\mathbf{N}\times[[\mathbf{v}]]$. Following Wu,[18] a dynamical equation for this strength γ can be written but will not be used here. For this inviscid fluid the thin shear layer is assumed to be without thickness and there is no viscosity ($\nu=0$) in Eq. (3) which becomes the Euler equation.

In local coordinates, Eqs. (5) and (6) become

$$-[[u]]+\frac{1}{\delta^t}\frac{\partial\delta^t}{\partial\varphi}[[v]]+\frac{1}{h_3}\left(\frac{\partial\delta^t}{\partial s}-\sigma T\frac{\partial\delta^t}{\partial\varphi}\right)[[w]]=0, \tag{7}$$

$$\frac{\partial\delta^t}{\partial t}-u+\frac{1}{\delta^t}\frac{\partial\delta^t}{\partial\varphi}v+\frac{1}{h_3}\left(\frac{\partial\delta^t}{\partial s}-\sigma T\frac{\partial\delta^t}{\partial\varphi}\right)w-\delta^t\dot{\mathbf{e}}_r\cdot\nabla F=0, \tag{8}$$

where all velocity components are taken on the interface $r=\delta^{t\pm}$.

C. Vortex ring bubble

In the case of an *inviscid* vortex ring bubble, the location of the free-boundary $r=\delta^b(\varphi,s,t)$ is another unknown function. The pressure jump at the free-boundary of the bubble is

$$[[p]]^b=p(\delta^{b+})-p(\delta^{b-})=2\Upsilon\kappa, \tag{9}$$

where Υ is the surface tension divided by the constant density of the incompressible fluid and $\kappa(\varphi,s,t)$ the mean curvature of the free-boundary.

The bubble[14] contains liquid vapor of pressure P_v and noncondensables of partial pressure $P_g=P_{g0}(\mathcal{V}_0^b/\mathcal{V}^b)^k$, where $\mathcal{V}^b$ is the volume of the bubble and k is the polytropic constant of the ideal gas in the bubble. Here, $\mathcal{V}_0^b$ and P_{g0} are their initial values. The pressure $p(\delta^{b-})=P_v+P_g$ inside the bubble is uniform. As pointed out by one of the referees, polytropic variations are merely an approximate simplification of the full thermodynamics as they are based on an ad hoc coupling between two otherwise independent thermodynamics variables. I will not remove this assumption of polytropic variations by including an energy balance equation in the analysis as was suggested by this referee and I postpone this work for the future.

The dynamical equation of the free-boundary is

$$\frac{\partial\delta^b}{\partial t}+(\mathbf{V}(\delta^{b+})-\delta^b\dot{\mathbf{e}}_r)\cdot\nabla F=0, \tag{10}$$

where ∇F is as before but with δ^t replaced by δ^b. For this inviscid fluid the thin diffusion layer along the interface is assumed to have no thickness and there is no viscosity ($\nu=0$) in Eq. (3) which becomes the Euler equation.

As the fluid is outside of such a bubble, singularities can exist inside the bubble, which is not possible in a homogeneous fluid, and the condition $u=v=0$ at $r=0$ is no longer valid. Vortex ring bubbles without axial variations and with a circular centerline have been studied by Genoux[14] for a potential flow. In fact, vortex ring bubble can be embedding in a vortical flow: for example, one can easily consider a vortex ring bubble with a vortex core of Rankine type. The circulation Γ of the ring is then $\Gamma=\Gamma_1+\Gamma_2$, where Γ_1 is the circulation induced by the vortex sheet on the free-boundary and Γ_2 is the added circulation due to the vortical core.

The Weber number is $W_e=L\Upsilon/(\varepsilon\Gamma^2)$ and I define $\overline{\Upsilon}=\varepsilon\Upsilon$, $\overline{P_v}=\varepsilon^2P_v$, $\overline{P_{g0}}=\varepsilon^2P_{g0}$. I am interested by the regime $W_e=O(\varepsilon^{-2})$ because the effect of the surface tension will come at leading order in the pressure jump (9).

D. Two-time analysis and expansions

In the two-time analysis, the expansion of the velocity $\dot{\mathbf{X}}=\partial_t\mathbf{X}+\varepsilon^{-1}\partial_\tau\mathbf{X}$ of the centerline is

$$\dot{\mathbf{X}}=\partial_t\mathbf{X}^{(0)}+\partial_\tau\mathbf{X}^{(1)}+O(\varepsilon\log\varepsilon),$$

with the following expansion of the centerline:

$$\mathbf{X}=\mathbf{X}^{(0)}(s,t)+\varepsilon\mathbf{X}^{(1)}(s,t,\tau=t/\varepsilon)+\dots.$$

Here, fast oscillations of the centerline of amplitude ε can exist.

The inner expansions of the relative velocity components and of the pressure are

$$u^{inn}=u^{(1)}(\overline{r},\varphi,s,t,\tau)+\dots,$$

$$v^{inn}=\varepsilon^{-1}v^{(0)}(\overline{r},s,t,\tau)+v^{(1)}(\overline{r},\varphi,s,t,\tau)+\dots,$$

$$w^{inn}=\varepsilon^{-1}w^{(0)}(\overline{r},s,t,\tau)+w^{(1)}(\overline{r},\varphi,s,t,\tau)+\dots,$$

$$p^{inn}=\varepsilon^{-2}p^{(0)}(\overline{r},s,t,\tau)+\varepsilon^{-1}p^{(1)}(\overline{r},\varphi,s,t,\tau)+\dots,$$

where $\overline{r}=r/\varepsilon$ is the stretched radial coordinate in the core. Here, the leading-order velocity field is axisymmetric, as in the previous asymptotic theories, but can change along the filament. There is no radial velocity at leading order. This is consistent with an axisymmetric leading order: e.g., an elliptic core at leading order will behave like a Kirchhoff ellipse which would rotate on the ultra-short time t/ε^2 and have a nonzero radial velocity at leading order.

Short wavelengths of order ε are out of the scope of this theory: their dynamics would be on the time $\overline{t}=t/\varepsilon^2$, would use a stretched axial coordinate $\overline{s}=s/\varepsilon$ and was studied by Widnall and Tsai[19] in the linear regime around a circular vortex ring.

If the vorticity is inside a vortex tube, the interface has the following expansion:

$$\overline{\delta^t}=\delta^t/\varepsilon=\overline{\delta^{t(0)}}(\varphi,s,t,\tau)+\varepsilon\overline{\delta^{t(1)}}(\varphi,s,t,\tau)+\ldots$$

and its time-derivative is

$$\dot{\delta}^t=\partial_\tau\overline{\delta^t}+\varepsilon\partial_t\overline{\delta^t}=\partial_\tau\overline{\delta^{t(0)}}+\varepsilon(\partial_\tau\overline{\delta^{t(1)}}+\partial_t\overline{\delta^{t(0)}})+\ldots.$$

For a vortex bubble, one have the same expansion for δ^b, and the expansions of the volume of the bubble and of the curvature of the free-boundary are

$$\frac{\mathcal{V}^{\,b}}{\varepsilon^2}=\mathcal{V}^{\,b(0)}+\varepsilon\mathcal{V}^{\,b(1)}+\ldots,$$

$$\kappa=\varepsilon^{-1}\kappa^{(0)}+\kappa^{(1)}+\ldots.$$

When the leading order is axisymmetric, it comes

$$\frac{\mathcal{V}^{\,b}}{\varepsilon^2}=\pi\int_0^{2\pi}\sigma^{(0)}[\overline{\delta^{b(0)}}]^2ds+\varepsilon\pi\int_0^{2\pi}(\sigma^{(1)}[\overline{\delta^{b(0)}}]^2+2\sigma^{(0)}\overline{\delta^{b(0)}}\,\overline{\delta^{cb(1)}})ds+O(\varepsilon^2),\tag{11}$$

$$\kappa=-\frac{1}{2\overline{\delta^{b(0)}}(s,t,\tau)}\frac{1}{\varepsilon}+\frac{1}{2}\frac{\overline{\delta^{b(1)}}(\varphi,s,t,\tau)}{[\overline{\delta^{b(0)}}(s,t,\tau)]^2}+\frac{1}{2}K^{(0)}\cos(\varphi)+\frac{1}{2}\frac{[\overline{\delta^{b(1)}}(\varphi,s,t,\tau)]_{\varphi\varphi}}{[\overline{\delta^{b(0)}}(s,t,\tau)]^2}+O(\varepsilon),\tag{12}$$

where $\overline{\delta^{cb(1)}}$ is the axisymmetric part of $\overline{\delta^{b(1)}}(\varphi,s,t,\tau)$.

III. TWO-TIME-SCALE DYNAMICS OF AXIAL VARIATIONS

The substitution of the previous expansions into Eqs. (2)–(3) leads to a cascade of asymptotic equations as in the one-time analysis.[1] In this section I give the two-time-scale equations for the dynamics of axial variations at leading order. It consists in the leading-order short-time axisymmetric equations in the filament [Eqs. (13)–(16)] and the leading-order short-time [Eq. (33)] and normal-time [Eq. (32)] equations of motion of the centerline. In the first subsection the leading-order equations of the short-time axisymmetric dynamics in the filament are given. The leading-order short-time asymmetric dynamics in the filament is slaved by the axisymmetric dynamics and is given in Appendix A. The motion of the centerline is slaved by this core dynamics and its induced velocity is given in the next subsection. Finally a qualitative description of the two-time dynamics in the filament is given.

A. The leading-order short-time axisymmetric dynamics of axial variations in the filament

The leading-order equations of the short-time axisymmetric dynamics in the filament come from the axisymmetric part at leading and first orders. I will give these equations in this subsection.

1. Continuous vorticity field

The leading-order compatibility conditions of the one-time analysis[1] become the dynamical equations of the axisymmetric part of the leading-order relative velocity field, i.e., of the axial core-variations.

a. The velocity form of the equations. At leading order

$$p^{(0)}=-\int_{\overline{r}}^{\infty}\frac{v^{(0)^2}}{\overline{r}}d\overline{r}+p^{(0)}(\infty),\tag{13}$$

and at first order

$$(\overline{r}u^{c(1)})_{\overline{r}}+\overline{r}w_z^{(0)}=0,\tag{14}$$

$$\frac{\partial v^{(0)}}{\partial\tau}+\zeta^{(0)}u^{c(1)}+w^{(0)}v_z^{(0)}=0,\tag{15}$$

$$\frac{\partial w^{(0)}}{\partial\tau}+w_r^{(0)}u^{c(1)}+p_z^{(0)}+w^{(0)}w_z^{(0)}=0,\tag{16}$$

where $\zeta^{(0)}=(\overline{r}v^{(0)})_{\overline{r}}/\overline{r}$ is the leading-order axial vorticity, $u^{c(1)}$ is the axisymmetric part of the radial velocity at first order, $p^{(0)}$ is the leading-order pressure, and $z=\int_0^s\sigma^{(0)}(s',t)ds'$.

This system for $p^{(0)}$, $v^{(0)}$, $w^{(0)}$, and $u^{c(1)}$ is closed. It gives the short-time-scale dynamics of the axial core-variations of axisymmetric shape. These equations are the same as the ones obtained by Souza[12] for a straight filament. They are the "long wave scaling" shallow water equations derived from studies of vortex breakdown of a straight filament.[20,12] Let us point out that in the studies of vortex breakdown and swirling-jets, the velocity field is often nondimensionalized using Γ/l, all lengths using l, and the time using l^2/Γ, where l is the small characteristic length and is of the thickness size. From this point of view, the $O(1)$ wavelength of the asymptotic theory of vortex motion is a long wavelength and the short wavelength of the Tsai and Widnall[19] study is a usual $O(1)$ wavelength.

At this order and on this short time the previous derivation shows that the curvature of the filament has no effect on the dynamics of axial variations. This proves the intuition of Souza who pointed out (private communication) that the equations for a straight filament might be relevant to a curved filament. For a curved filament they were first given in Margerit and Brancher.[13]

I will now give other useful forms of these equations.

b. The stream-function form of the equations. Let us define the meridional stream function $\psi^{c(1)}$, with

$$u^{c(1)}=-\frac{1}{\overline{r}}\psi_z^{c(1)},\quad w^{(0)}=\frac{1}{\overline{r}}\psi_{\overline{r}}^{c(1)},$$

and introduce the following transformation:

$$\mathcal{K}^{(0)}=\overline{r}v^{(0)},\quad y=\overline{r}^2,$$

usually used to study these equations.[20] In these new variables the previous system becomes

$$\frac{\partial \mathcal{K}^{(0)}}{\partial \tau} - 2\psi_z^{c(1)}\mathcal{K}_y^{(0)} + 2\psi_y^{c(1)}\mathcal{K}_z^{(0)} = 0, \tag{17}$$

$$D^2\frac{\partial \psi^{c(1)}}{\partial \tau} + 2\psi_y^{c(1)}D^2\psi_z^{c(1)} + \frac{2}{y}\mathcal{K}^{(0)}\mathcal{K}_z^{(0)} - 2y\psi_z^{c(1)} \times [y^{-1}D^2\psi^{c(1)}]_y = 0, \tag{18}$$

where $D^2\psi^{c(1)} = 4y\psi_{yy}^{c(1)}$.

c. The Souza form of the equations. The previous equations can also be written in the form[12]

$$\frac{\partial \mathcal{K}^{(0)}}{\partial \tau} + \frac{1}{\bar{r}}\nabla^{\perp}\psi^{c(1)} \cdot \nabla \mathcal{K}^{(0)} = 0, \tag{19}$$

$$L\psi^{c(1)} = \bar{r}^2 \xi^{(0)}, \tag{20}$$

$$\frac{\partial \xi^{(0)}}{\partial \tau} + \frac{1}{\bar{r}}\nabla^{\perp}\psi^{c(1)} \cdot \nabla \xi^{(0)} + \frac{2}{\bar{r}^4}\mathcal{K}^{(0)}\mathcal{K}_z^{(0)} = 0, \tag{21}$$

where $\nabla = (\partial_{\bar{r}}, \partial_z)$, $\nabla^{\perp} = (-\partial_z, \partial_{\bar{r}})$, $L = \partial_{\bar{r}}^2 - \partial_{\bar{r}}/\bar{r}$, and $-\bar{r}\xi^{(0)} = -w_{\bar{r}}^{(0)}$ is the leading-order circumferential (azimuthal) vorticity.

The boundary conditions to these equations are the periodicity between $z=0$ and $z=S^{(0)}$, where $S^{(0)}$ is the length of the closed vortex, $\psi_{\bar{r}}^{c(1)}/\bar{r} \to 0$ and $\mathcal{K}^{(0)} \to \Gamma/2\pi$ at infinity, and $\partial\psi^{c(1)}/\partial\bar{r} = 0$ and $\partial\mathcal{K}^{(0)}/\partial\bar{r} = 0$ at $\bar{r}=0$.

d. The equations in the Von Mises variables. One can replace the independent variables $(\tau, \bar{r}, s)$ by $(\tau, \psi^{c(1)}, s)$ at all point where $w^{(0)} \neq 0$. In these Von Mises variables, the previous system becomes the nonstationary Bragg–Hawthorne (or Squire–Long) equation

$$L\psi^{c(1)} = \bar{r}^2 \frac{\partial \mathcal{H}^{(0)}}{\partial \psi^{c(1)}} - \mathcal{K}^{(0)}\frac{\partial \mathcal{K}^{(0)}}{\partial \psi^{c(1)}} \tag{22}$$

where $\mathcal{H}^{(0)} = p^{(0)} + (v^{(0)^2} + w^{(0)^2})/2$.

2. Discontinuous vorticity field

For a vortex with the vorticity inside a vortex tube the interface dynamics has to be found and the previous equations have to be completed. The leading order of the condition of continuity (4) of the pressure on the interface yields

$$[[p^{(0)}]] = 0. \tag{23}$$

Here and in the following, I use the notation

$$[[f]] \equiv f(\bar{\delta}^{t(0)+}) - f(\bar{\delta}^{t(0)-})$$

for the jump on the interface. This continuity of the leading-order pressure means that the expression (13) for $p^{(0)}$ is correct even through the interface. The first order of the continuity of the normal velocity (7) yields

$$[[u^{c(1)}]] = \bar{\delta}_z^{t(0)}[[w^{(0)}]], \tag{24}$$

which can be written for $\psi^{c(1)}$ as

$$[[\psi_z^{c(1)}]] = -\bar{\delta}^{t(0)}\bar{\delta}_z^{t(0)}[[w^{(0)}]]. \tag{25}$$

The axisymmetric part of the kinematic boundary condition (6) at first order gives

$$\frac{\partial \bar{\delta}^{t(0)}}{\partial \tau} - u^{c(1)} + w^{(0)}\bar{\delta}_z^{t(0)} = 0, \tag{26}$$

where all velocity components are taken on the interface $\bar{r} = \bar{\delta}^{t(0)\pm}$.

This system for $p^{(0)}$, $v^{(0)}$, $w^{(0)}$, $u^{c(1)}$, and $\bar{\delta}^{t(0)}$ is closed.

3. Vortex ring bubble

For a vortex bubble the interface dynamics has to be found and the previous equations have to be completed. The leading-order of the axisymmetric part of the pressure jump (9) is

$$p^{(0)}(\bar{\delta}^{b(0)+}) = \left(\frac{\Gamma}{2\pi}\right)^2 \frac{C_p(s,\tau,t)}{(\bar{\delta}^{b(0)})^2} + p^{(0)}(\infty)$$
$$= \bar{P}_v + \bar{P}_{g0}\left(\frac{\mathcal{V}_0^{b(0)}}{\mathcal{V}^{b(0)}}\right)^k - \frac{\bar{\Upsilon}}{\bar{\delta}^{b(0)}}, \tag{27}$$

where

$$C_p(s,\tau,t) = -\left(\frac{2\pi\bar{\delta}^{b(0)}}{\Gamma}\right)^2 \int_{\bar{\delta}^{b(0)}}^{\infty} \frac{v^{(0)^2}(\bar{r},s,\tau,t)}{\bar{r}} d\bar{r}.$$

Equation (27) is the equation of the thickness $\bar{\delta}^{b(0)}$ of the bubble.

The leading-order for axisymmetric part of the dynamical equation of the free-boundary (10) is

$$\frac{\partial \bar{\delta}^{b(0)}}{\partial \tau} - u^{c(1)} + w^{(0)}\bar{\delta}_z^{b(0)} = 0, \tag{28}$$

where all velocity components are taken on the free-boundary $\bar{r} = \bar{\delta}^{b(0)+}$. The bubble allows to have a solution of the equation of continuity (14) in the form

$$u^{c(1)} = \frac{\mathcal{D}^{c(1)}(s,\tau,t)}{\bar{r}} + u^{\omega c(1)},$$

where $u^{\omega c(1)}$ is regular at $\bar{r}=0$. As the thickness $\bar{\delta}^{b(0)}$ is given by Eq. (27), Eq. (28) is indeed the equation for $\mathcal{D}^{c(1)}$.

This system for $p^{(0)}$, $v^{(0)}$, $w^{(0)}$, $u^{\omega c(1)}$, $\mathcal{D}^{c(1)}$, and $\bar{\delta}^{b(0)}$ is closed.

B. The two-time-scale dynamics of the centerline

In the previous subsection the short-time-scale dynamical equations of the axial core-variations were given for the velocity field in the core. In this subsection the dynamical equation of the induced velocity of the centerline is given. It comes from the matching law of the inner and outer velocity fields. This motion of the centerline is slaved by the leading-order core dynamics.

1. Combined form of the equation

At leading order the outer velocity is

$$\mathbf{v}(\mathbf{x}) = \frac{\Gamma}{4\pi}\int \frac{\sigma^{(0)}(s,t)\mathbf{t}(s,t)(\mathbf{x}-\mathbf{X}^{(0)}(s,t))}{|\mathbf{x}-\mathbf{X}^{(0)}(s,t)|^3}\,ds. \qquad (29)$$

This is the velocity induced by the vorticity $\boldsymbol{\omega}=\delta(\mathbf{x}-\mathbf{X}^{(0)})\mathbf{t}$ concentrated on the leading-order centerline $\mathbf{X}^{(0)}$, where $\delta(\mathbf{x}-\mathbf{X}^{(0)})$ is the delta-function on $\mathbf{X}^{(0)}(s,t)$. The axial core-variations have no leading-order effect in the outer region. This results of the assumption that fast oscillations of the centerline are of amplitude ε as stated in the form of the expansion of $\mathbf{X}$ in Sec. II.

At first order the matching law between this outer solution and the inner solution (found in Sec. III A, and Appendix A) yields

$$\partial_t\mathbf{X}^{(0)}(s,t)+\partial_\tau\mathbf{X}^{(1)}(s,\tau,t) = \mathbf{A}(s,t)+\Gamma\frac{K^{(0)}(s,t)}{4\pi}\left[\log\left(\frac{S^{(0)}}{\varepsilon}\right)-1+C_v+C_w\right]\mathbf{b}(s,t), \qquad (30)$$

where

$$\mathbf{A}(s,t)=\frac{\Gamma}{4\pi}\int_{-\pi}^{+\pi}\mathbf{a}\,ds',$$

$$\mathbf{a}=\sigma^{(0)}(s+s',t)\left[\frac{\mathbf{t}(s+s',t)\times\mathbf{g}}{|\mathbf{g}|^3}-\frac{K^{(0)}(s,t)\mathbf{b}(s,t)}{2|\lambda(s,s',t)|}\right],$$

$$\mathbf{g}=\mathbf{X}^{(0)}(s,t)-\mathbf{X}^{(0)}(s+s',t),$$

and $\lambda(s,s',t)=\int_s^{s+s'}\sigma^{(0)}(s^*,t)ds^*$. In this Eq. (30), $C_v(s,\tau,t)$ and $C_w(s,\tau,t)$ are known functions, which describe the circumferential and axial evolution of the inner velocity in the core:

$$C_v(s,\tau,t)=\frac{1}{2}+\lim_{\bar r\to+\infty}\left(\frac{4\pi^2}{\Gamma^2}\int_0^{\bar r}\overline{r'}\,v^{(0)^2}(\overline{r'},s,\tau,t)d\overline{r'} - \log\bar r\right),$$

$$C_w(s,\tau,t)=-\frac{1}{2}\left(\frac{4\pi}{\Gamma}\right)^2\int_0^\infty \bar r\,w^{(0)^2}(\bar r,s,\tau,t)d\bar r.$$

Equation (30) extends the Callegari and Ting[1] equation of vortex filament motion to axial core-variations of the leading-order velocity field. It holds both for a continuous or a discontinuous vorticity field.

Any initial condition that does not satisfy the induced asymmetric flow field [Eq. (A8) in Appendix A], and Eq. (30) will need a three-time-scale analysis $(\bar t=t/\varepsilon^2,\tau,t)$. These small-amplitude oscillations of order ε^2 have already been introduced by Ting and Tung[21] and Gunzburger[22] to study a straight vortex filament with an initial velocity that is different from the potential background velocity on the filament. An example of such a curved filament, that does not satisfy the induced asymmetric flow field [Eq. (A8) in Appendix A], is given in Margerit and Brancher.[13]

The regime studied in this section is not the same as the one considered by Ting and Klein,[23] who studied axial core-variations on an *open* vortex filament by means of a single-time-scale t and double-axial-scale $(s,\tilde\xi=\varepsilon s)$ analysis. For an *open* filament, the double-time-scale analysis $(t,\tau=t/\varepsilon)$ induces that the normal-time t behavior of a core-variation perturbation, which evolves at short-time-scale $\tau=t/\varepsilon$, is to reach the far-distance $\tilde\xi=\varepsilon s$ of the Ting and Klein[23] regime. Thus, except if the open filament is periodical, a double-time-scale analysis $(t,\tau=t/\varepsilon)$ coupled to a double-axial-scale $(s,\tilde\xi=\varepsilon s)$ analysis would be needed to describe the dynamics of the open filament.

For a vortex bubble, the matching law between the outer solution and the inner solution yields the equation

$$\partial_t\mathbf{X}^{(0)}(s,t)+\partial_\tau\mathbf{X}^{(1)}(s,\tau,t) = \mathbf{A}(s,t)+\Gamma\frac{K^{(0)}(s,t)}{4\pi}\left[\log\left(\frac{S^{(0)}}{\varepsilon}\right)-1+C_v+C_w+\overline{W}_e\right]\mathbf{b}(s,t), \qquad (31)$$

where $\overline{W}_e(s,\tau,t)=4\pi^2\overline{\delta}^{b(0)}(s,\tau,t)\overline{\Upsilon}/\Gamma^2$. Here, the inner velocity field of Sec. III A and Appendix A was used. The adding term in Eq. (31) as regard of Eq. (30) is due to the difference between the inner velocity fields.

2. Time averaging and splitting form of the equation

The τ-average of a function $f(\tau,t)$, is denoted $\mathcal{M}f$, and is defined by

$$\mathcal{M}f=\lim_{\tilde T\to+\infty}\frac{1}{T}\int_{\varepsilon^{-1}t}^{\varepsilon^{-1}t+\tilde T}f(\tau,t)d\tau,$$

where $\tilde T$ is an intermediate variable: $\tau\ll\tilde T\ll t$. The τ-average of Eq. (30) yields the leading-order equation of motion of the filament in the normal-time scale

$$\partial_t\mathbf{X}^{(0)}(s,t)=\mathbf{A}(s,t)+\Gamma\frac{K^{(0)}(s,t)}{4\pi}\left[\log\left(\frac{S^{(0)}}{\varepsilon}\right)-1+\mathcal{M}C_v+\mathcal{M}C_w\right]\mathbf{b}(s,t), \qquad (32)$$

where

$$\mathcal{M}C_v=\frac{1}{2}+\lim_{\bar r\to+\infty}\left(\frac{4\pi^2}{\Gamma^2}\int_0^{\bar r}\overline{r'}\mathcal{M}(v^{(0)^2})d\overline{r'}-\log\bar r\right),$$

$$\mathcal{M}C_w=-\frac{1}{2}\left(\frac{4\pi}{\Gamma}\right)^2\int_0^\infty\bar r\mathcal{M}(w^{(0)^2})d\bar r.$$

The subtraction of Eq. (32) from (30) leads to the equation for $\mathbf{X}^{(1)}$ in the short-time scale:

$$\partial_\tau\mathbf{X}^{(1)}(s,\tau,t)=\frac{\Gamma K^{(0)}(s,t)}{4\pi}(\Delta C_v+\Delta C_w)\mathbf{b}(s,t), \qquad (33)$$

where

$$\Delta C_v = \frac{4\pi^2}{\Gamma^2}\int_0^\infty \overline{r}[v^{(0)^2} - \mathcal{M}(v^{(0)^2})]d\overline{r},$$

$$\Delta C_w = \frac{1}{2}\left(\frac{4\pi}{\Gamma}\right)^2 \int_0^\infty \overline{r}[w^{(0)^2} - \mathcal{M}(w^{(0)^2})]d\overline{r}.$$

The first-order equation of motion of the filament in the normal-time scale, i.e., the equation for $\partial_t \mathbf{X}^{(1)}$, would come from the matching at next order. An important result stated by Eq. (33) is that the fast oscillations of small ε-amplitude of the centerline are only in the binormal direction and are proportional to the local curvature.

C. The two-time-scale dynamics of axial variations in the filament

The previous subsection shows that we need to know the normal-time average $\mathcal{M}(v^{(0)^2})$ and $\mathcal{M}(w^{(0)^2})$ of the square of the velocity components to find the normal-time evolution of the centerline. I will now give qualitative ideas of the full two-time-scale dynamics.

Let us first assume that $\mathcal{M}(v^{(0)^2}) = \mathcal{M}(v^{(0)})^2$ and that the axial velocity also satisfies this property. Let us have the conjecture that core equations (13)–(16) give the dynamics of the short-time variations around this averaged state $\mathcal{M}(v^{(0)})$ but do not give the dynamics of this averaged state and that these variations are bounded. We need to look at the axisymmetric part of the equations at second order to extract the needed dynamical equations of the averaged state.

The second-order equations [Eqs. (B1)–(B4) in Appendix B] is a linear system of equations for $p^{c(1)}$, $v^{c(1)}$, $w^{c(1)}$, and $u^{c(2)}$. This system gives the dynamics of the first-order axisymmetric axial variations. It has inhomogeneous terms and nonconstant coefficients, which depend only on the leading-order velocity field is satisfied. Let us also have the second conjecture that this linear operator is not uniquely invertible. The Fredholm alternative implies that this inhomogeneous linear system of equations has bounded solutions only if a compatibility condition for the leading-order velocity field. This compatibility condition is the dynamical equation of the leading-order time-averaged state.

More theoretical and numerical works have to be done to prove these two conjectures. I will not do this work in this paper. Nevertheless, as a first step, Sec. IV gives one-time solutions of the equations and Sec. V studies the two-time-scale dynamics of small axial variations around these one-time-scale solutions. The study of this linearized leading-order operator may also help to study the linear system of equations for $p^{c(1)}$, $v^{c(1)}$, $w^{c(1)}$, and $u^{c(2)}$ and to carry out its Fredholm alternative.

IV. THE ONE-TIME FILAMENT SOLUTIONS

In this section I give solutions to the one-time equations. These solutions will be used in Sec. V to study the two-time-scale dynamics of small axial variations around this base flow.

If the short-time scale derivative is removed from Eqs. (13)–(16) or from the equivalent equations (17)–(18), these equations become compatibility equations for the one-time solution at leading order, which may be called a quasi-stationary solution of these equations. Leading-order velocity fields without axial variations and $u^{c(1)} = 0$ are solutions of these leading-order compatibility conditions.

Appendix C considers the uniqueness of these compatibility conditions. The study of small perturbations around the solutions without axial variations seems to indicate that they are the unique solutions to these compatibility conditions. Appendix D considers the uniqueness of the compatibility conditions at second order. Assuming that the compatibility conditions at first order have the only solution without axial variations it is found that these compatibility conditions at second order also have a unique solution, which is given. This solution is the one introduced by Margerit[17] to generalize the Callegari and Ting theory[1] at next order.

A. The one-time equations

I now consider the leading-order velocity fields without axial variations. For a closed filament, the s-average of quasi-stationary (one-time) solutions of axisymmetric equations at second order [Eqs. (B1)–(B4) in Appendix B] satisfies[1]

$$\frac{\partial v^{(0)}}{\partial t} - \overline{\nu}\zeta_{\overline{r}}^{(0)} = \frac{1}{2}\overline{r}\zeta^{(0)}\frac{\dot{S}^{(0)}}{S^{(0)}}, \tag{34}$$

$$\frac{\partial w^{(0)}}{\partial t} - \overline{\nu}\frac{1}{\overline{r}}[\overline{r}w_{\overline{r}}^{(0)}]_{\overline{r}} = \frac{1}{2}\overline{r}^3\left(\frac{w^{(0)}}{\overline{r}^2}\right)_{\overline{r}}\frac{\dot{S}^{(0)}}{S^{(0)}}, \tag{35}$$

where the leading-order quasi-stationary velocity is without axial variations and $u^{c(1)} = 0$ as previously stated. Equations (32), (34), and (35) derived by Callegari and Ting[1] are a complete set of equations for the one-time solution, which is without axial variations. This *one-time* solution is in some sense the generalization to vortex filaments with centerline of any shape of the *stationary* circular vortex ring solution in a translative frame.

B. The one-time solutions in dimensionless form

Callegari and Ting[1] used a special transformation to find the solutions of Eqs. (32), (34), and (35). In the remaining of this section the core-function $C_v(t)$ and $C_w(t)$ are given and displayed in a simple way. These expressions of the core functions and Eq. (32) are a complete set of equations for the one-time motion of the centerline of the filament.

Let us define the following similarity functions

$$v^{*(0)} = v^{(0)}\overline{\delta}/\Gamma,$$

$$\zeta^{*(0)} = \zeta^{(0)}\overline{\delta}^2/\Gamma,$$

$$w^{*(0)} = w^{(0)}\frac{\overline{\delta}^2}{\Gamma\overline{\delta}_0}\left(\frac{S_0^{(0)}}{S^{(0)}(t)}\right)^{-2},$$

$$\mathcal{K}^{*(0)} = \mathcal{K}^{(0)}/\Gamma,$$

$$\psi^{*c(1)} = \psi^{c(1)}\frac{1}{\Gamma\overline{\delta}_0}\left(\frac{S_0^{(0)}}{S^{(0)}(t)}\right)^{-2},$$

where $\overline{\delta_0}$, $S_0^{(0)}$ are the initial thickness and length of the vortex and $\overline{\delta}$, $S^{(0)}$ their values at time t.

1. Inviscid fluid

If the fluid is inviscid ($\overline{\nu}=0$) the solutions are in the form[24]

$$v^{*(0)}=v_0^*(\eta),$$

$$w^{*(0)}=w_0^*(\eta),$$

where $\eta=\overline{r}/\overline{\delta}$, $(v_0^*(\eta),w_0^*(\eta))$ are the initial velocity fields, and $\overline{\delta}(t)$ is the ε-stretched thickness of the core

$$\overline{\delta}^2(t)=\overline{\delta}_0^2(S_0^{(0)}/S^{(0)}(t)),$$

where $S_0^{(0)}$ is the initial length of the filament. The associated core functions are

$$C_v(t)=C_v(0)-\log(\overline{\delta}(t)/\overline{\delta_0}),$$

$$C_w(t)=C_w(0)(S_0^{(0)}/S^{(0)}(t))^3,$$

where $C_v(0)$ and $C_w(0)$ are the associated initial core constant:

$$C_v(0)=\frac{1}{2}+\lim_{\eta\to+\infty}\left(4\pi^2\int_0^\eta \eta' v_0^{*2}(\eta')d\eta'-\log\eta\right)-\log\overline{\delta_0},$$

$$C_w(0)=-2\pi^2\int_0^\infty \eta w_0^{*2}(\eta)d\eta.$$

2. Viscous fluid

If the flow is viscous ($\overline{\nu}\neq 0$) the solutions of Eqs. (34)–(35) and (30) are in the form[25]

$$v^{*(0)}=\frac{1}{\eta}\left[\frac{1}{2\pi}(1-e^{-\eta^2})+e^{-\eta^2}\sum_{n=1}^{\infty}D_n^*P_n(\eta^2)1_{\overline{\nu}}^{-n}\right],$$

$$w^{*(0)}=\left[\frac{S_w(0)}{\pi}e^{-\eta^2}+2e^{-\eta^2}\sum_{n=1}^{\infty}C_n^*L_n(\eta^2)1_{\overline{\nu}}^{-n}\right],$$

where $\eta=\overline{r}/\overline{\delta}$, and $\overline{\delta}(t)$ is the ε-stretched thickness of the core

$$\overline{\delta}^2(t)=\overline{\delta}_0^2\left(\frac{S_0^{(0)}}{S^{(0)}(t)}\right)1_{\overline{\nu}},$$

$$1_{\overline{\nu}}=1+\frac{\overline{\delta}_{\overline{\nu}}^2}{\overline{\delta}_0^2},$$

$$\overline{\delta}_{\overline{\nu}}^2=4\overline{\nu}\int_0^t\frac{S^{(0)}(t')}{S_0^{(0)}}dt'.$$

Here, L_n are the Laguerre polynomials, $P_n(\eta^2)=L_{n-1}(\eta^2)-L_n(\eta^2)$, γ is the Euler's constant, $\overline{\delta}_{\overline{\nu}}$ is the diffusion-added ε-stretched thickness of the core, and (C_n^*,D_n^*) are the Fourier components of the initial axial velocity and tangential vorticity

$$C_n^*=\int_0^\infty w_0^*(\eta)L_n(\eta^2)\eta\,d\eta,$$

$$D_n^*=\int_0^\infty \zeta_0^*(\eta)L_n(\eta^2)\eta\,d\eta,$$

$$C_0^*=S_w(0)/2\pi,$$

$$D_0^*=1/2\pi.$$

$S_w(0)=m_0/(\Gamma\overline{\delta_0})$ is the initial swirl number, where m_0 is the initial axial flux. The swirl number $S_w(t)$ at time t is defined by $S_w(t)=m(t)/(\Gamma\overline{\delta})$, where $m(t)$ is the axial flux at time t. The associated core functions are

$$C_v(t)=-\log\overline{\delta}+\frac{1}{2}(1+\gamma-\log 2)+4\pi^2\sum_{(n,m)\in\mathbb{N}^2\setminus(0,0)}\frac{D_n^*D_m^*A_{nm}}{n+m}1_{\overline{\nu}}^{-(n+m)},$$

$$C_w(t)=-2\left(\frac{\overline{\delta_0}}{\overline{\delta}}\right)^2\left(\frac{S_0^{(0)}}{S^{(0)}(t)}\right)^4\left[S_w^2(0)+8\pi^2\sum_{(n,m)\in\mathbb{N}^2\setminus(0,0)}C_n^*C_m^*A_{nm}1_{\overline{\nu}}^{-(n+m)}\right],$$

where

$$A_{nm}=\int_0^\infty e^{-2x}L_n(x)L_m(x)dx=\frac{(n+m)!}{n!m!2^{m+n+1}}.$$

In the inviscid limit $\overline{\nu}\to 0$, we recover the inviscid velocity field previously given. This clearly shows the continuity of the analyses for the Navier–Stokes and Euler equations in the asymptotic ansatz based on the small slenderness ratio.

3. Similar vortex core

For a viscous *similar vortex core*[1]

$$v^{*(0)}=\frac{1}{2\pi\eta}(1-e^{\eta^2}),$$

$$w^{*(0)}=\frac{S_w(0)}{\pi}e^{-\eta^2},$$

$$C_v(t)=-\log\overline{\delta}(t)+\frac{1}{2}(1+\gamma-\log 2),$$

$$C_w(t)=-2\left(\frac{\overline{\delta_0}}{\overline{\delta}}\right)^2\left(\frac{S_0^2}{S^{(0)}(t)}\right)^4 S_w^2(0).$$

The relative velocity field of this similar vortex also depends only on one parameter: the initial swirl number $S_w(0)$ and is independent of any parameter if the axial velocity $w^{*(0)}$ is divided by this parameter.

4. Discontinuous vorticity field

For a vortex with the vorticity inside a vortex tube, the s-average of quasi-stationary (one-time) solutions of the axisymmetric part of the kinematic boundary condition at second order [Eq. (B6) in Appendix B] satisfies

$$\dot{\overline{\delta}}^{t(0)} = -\frac{1}{2}\overline{\delta}^{t(0)}\frac{\dot{S}^{(0)}}{S^{(0)}}. \tag{36}$$

Here, I used $\langle \sigma^{(0)} u^{c(2)} \rangle = -\overline{r}\dot{S}^{(0)}/2$, where $\langle \quad \rangle$ denotes the s-average, as can be found from the s-average of the axisymmetric part of the continuity equation at second order [Eq. (B2) in Appendix B]. It comes

$$(\overline{\delta}^{t(0)}(t))^2 = (\overline{\delta}_0^{t(0)})^2 (S_0^{(0)}/S^{(0)}(t)). \tag{37}$$

This equation is coherent with the function $\overline{\delta}^2(t) = \overline{\delta}_0^2 (S_0^{(0)}/S^{(0)}(t))$ previously introduced to describe the thickness of the continuous vorticity field in an inviscid fluid ($\overline{\nu}=0$). In this case I choose the thickness $\overline{\delta}$ of the vortex to be the interface thickness $\overline{\delta}^t$. For a Rankine vortex core with a uniform axial jet:

$$v_0^*(\eta) = \begin{cases} \dfrac{\eta}{2\pi} \\ \dfrac{1}{2\pi\eta} \end{cases}, \quad w_0^*(\eta) = \begin{cases} S_w(0)/\pi & \text{if } \eta < 1 \\ 0 & \text{if } \eta > 1. \end{cases} \tag{38}$$

It comes $C_v(t) = 3/4 - \log \overline{\delta}$ and $C_w(0) = -4 S_w(0)^2$. The relative velocity field of this Rankine vortex core with a uniform axial jet depends only on one parameter: the initial swirl number $S_w(0)$ and is independent of any parameter if the axial velocity $w^{*(0)}$ is divided by this parameter.

V. TWO-TIME-SCALE DYNAMICS OF AXIAL VARIATIONS IN THE SMALL AMPLITUDE LIMIT

In this section, I consider small axisymmetric axial variations around the one-time scale solutions (32), (34), and (35), which is the base flow. The leading-order equations of these perturbations will be found as a linearization near this base flow of the double-time-scale equations for the core (17), (18), and for the filament motion (32)–(33). This gives the equations [Eqs. (39)–(41)] of the dynamics of the small axial variations around the one-time base flow. From these equations the eigenvalue equations for linear Fourier modes [Eqs. (59)–(61)] are given for the stream function. This eigenvalue problem is then solved for both a Rankine and a similar core.

This study is more general than previous ones because the vortex filament of the base flow is not restricted to be circular[26,27,9] nor straight.[28–31] In the studies[28–30] of the stability of straight vortex filament, the characteristic length-scale that is used is the one of the thickness of the filament and so the long-wavelength limit has to be carried out to obtain our $O(1)$ wavelength regime, in which the characteristic length-scale that is used is the one of the radius of curvature of the filament.

This study is a linear stability analysis in the small thickness ε limit and in the moving frame of the perturbed flow. The coordinates are local coordinates in this frame and are not local coordinates to the base flow. The perturbation, as the base flow, is split into the relative velocity field and the filament motion. In the previous analysis the perturbation that is used is the one of the absolute velocity field and is the superposition of axisymmetric modes $e^{ins}e^{i\omega\tau}$ and bending modes $e^{\pm i\varphi}e^{ins}e^{i\omega\tau}$, where the coordinates are local to the base flow. In the case of a circular vortex ring, the results of my analysis can be derived from the one of Kopiev and Chernyshev[9] by: (i) deriving the velocity field from their displacement field; (ii) writing this field in the usual coordinates $(\overline{r},\varphi,s)$ of the moving frame; and (iii) splitting the velocity perturbation into the relative velocity and the filament velocity. I prefer to derive it in a simpler and straightforward way as follows.

A. Base flow and small perturbations

I introduce a small axisymmetric perturbation

$$(\tilde{u}^{c(1)},\tilde{v}^{(0)},\tilde{w}^{(0)},\tilde{p}^{(0)},\tilde{\psi}^{c(1)},\tilde{\mathcal{K}}^{(0)},\tilde{\mathbf{X}}^{(0)},\tilde{\mathbf{X}}^{(1)})$$

of a one-time flow without axial variations, denoted by $(\underline{u}^{c(1)}=0,\underline{v}^{(0)},\underline{w}^{(0)},\underline{p}^{(0)},\underline{\psi}^{c(1)},\underline{\mathcal{K}}^{(0)},\mathbf{X}^{(0)},\mathbf{X}^{(1)}=0)$:

$$\psi^{c(1)} = \underline{\psi}^{c(1)} + \mu\tilde{\psi}^{c(1)},$$

$$\mathcal{K}^{(0)} = \underline{\mathcal{K}}^{(0)} + \mu\tilde{\mathcal{K}}^{(0)},$$

$$\mathbf{X}^{(1)} = \mu\tilde{\mathbf{X}}^{(1)}.$$

For a vortex with the vorticity inside a vortex tube, the interface function is also unknown. Its perturbation and the one of the pressure are given by

$$\overline{\delta}^{t(0)} = \underline{\overline{\delta}}^{t(0)} + \mu\tilde{\overline{\delta}}^{t(0)},$$

$$p^{(0)} = \underline{p}^{(0)} + \mu\tilde{p}^{(0)}.$$

The base flow is the one-time scale solution given in Sec. IV and is without axial variations. Here, one has to restrict the form of the perturbations to the axisymmetric axial variations, for they are the perturbations we are interested by. In that sense I will not consider normal-time perturbations without axial variations of the relative velocity field or of the centerline. This induces the two following assumptions. The perturbations of the relative velocity field have axial core-variations and are assumed to have null axial average. With this assumption and with the uniqueness study of Appendix C we deduce that these perturbations have no normal-time-scale dynamics, i.e., $\mathcal{M}(\tilde{v}^{(0)})=0$ and $\mathcal{M}(\tilde{w}^{(0)})=0$. Moreover as the motion of the centerline at leading order $\mathbf{X}^{(0)}$ is a normal-time-scale dynamics I assume that the leading-order centerline is not perturbed, i.e., $\tilde{\mathbf{X}}^{(0)}=0$. These two assumptions are not restrictive; they only means that in the perturbation we do not have the bending modes of the one-time scale, which have already been studied elsewhere.[10,11]

B. Two-time-scale linear equations of small axial variations

1. Continuous vorticity field

From Eqs. (17)–(18) we deduce that at first order in the small amplitude μ the perturbation of the relative velocity field satisfies

$$\frac{\partial \tilde{\mathcal{K}}^{(0)}}{\partial \tau} - 2\underline{\mathcal{K}}_y^{(0)}\frac{\partial \tilde{\psi}^{c(1)}}{\partial z} + 2\underline{\psi}_y^{c(1)}\frac{\partial \tilde{\mathcal{K}}^{(0)}}{\partial z} = 0, \tag{39}$$

$$D^2\frac{\partial \tilde{\psi}^{c(1)}}{\partial \tau} + \frac{2}{y}\underline{\mathcal{K}}^{(0)}\frac{\partial \tilde{\mathcal{K}}^{(0)}}{\partial z} + \mathcal{G}\left(\frac{\partial \tilde{\psi}^{c(1)}}{\partial z}\right) = 0, \tag{40}$$

where

$$\mathcal{G} = 2\underline{\psi}_y^{c(1)}D^2 - 8y\underline{\psi}_{yyy}^{c(1)}.$$

These equations give the short-time dynamics of the small amplitude axial variations in the filament.

At first order in μ the τ-averages $\mathcal{M}(v^{(0)^2})$ and $\mathcal{M}(w^{(0)^2})$ are given by $\mathcal{M}(v^{(0)^2}) = \mathcal{M}(\underline{v}^{(0)^2} + 2\underline{v}^{(0)}\tilde{v}^{(0)}) = \underline{v}^{(0)^2} + 2\underline{v}^{(0)}\mathcal{M}(\tilde{v}^{(0)}) = \underline{v}^{(0)^2}$, as $\mathcal{M}(\tilde{v}^{(0)}) = 0$, and $\mathcal{M}(w^{(0)^2}) = \underline{w}^{(0)^2}$. This means that the small perturbations of the relative velocity field have no normal-time-scale dynamics as it has previously been assumed. From Eqs. (32) we can check that the leading-order centerline is not perturbed, i.e., $\tilde{\mathbf{X}}^{(0)} = 0$, as it has previously been assumed.

From Eqs. (33) we deduce that the perturbation of the filament velocity satisfies

$$\partial_\tau \tilde{\mathbf{X}}^{(1)}(s,\tau,t) = \frac{\Gamma K^{(0)}(s,t)}{4\pi}(\Delta C_v + \Delta C_w)\mathbf{b}(s,t), \tag{41}$$

where

$$\Delta C_v = \frac{4\pi^2}{\Gamma^2}\int_0^\infty 2\overline{r}\underline{v}^{(0)}\tilde{v}^{(0)}\, d\overline{r},$$

$$\Delta C_w = \frac{1}{2}\left(\frac{4\pi}{\Gamma}\right)^2\int_0^\infty 2\overline{r}\underline{w}^{(0)}\tilde{w}^{(0)}\, d\overline{r}.$$

This shows that for a curved vortex filament the perturbations with axial variations induce small oscillations of amplitude ε of the centerline and that these perturbations are in the binormal direction and proportional to the curvature. The system of Eqs. (39)–(41) gives the dynamics of the small axial variations around the one-time base flow.

2. Discontinuous vorticity field

For a vortex with the vorticity inside a vortex tube, the continuity of the leading-order pressure (23) becomes

$$[\![\tilde{p}^{(0)}]\!] = -\frac{\tilde{\delta}^{t(0)}}{\overline{\underline{\delta}}^{t(0)}}[\![\underline{v}^{(0)^2}]\!], \tag{42}$$

where I used Eq. (13) to have $[\![\underline{p}_r^{(0)}]\!] = [\![\underline{v}^{(0)^2}]\!]/\overline{\delta}^{t(0)}$. Here and in the following I use the notation

$$[\![f]\!] \equiv f(\overline{\underline{\delta}}^{t(0)+}) - f(\overline{\underline{\delta}}^{t(0)-})$$

for the jump on the interface. The continuity of the normal velocity (25) becomes

$$[\![\tilde{\psi}_z^{c(1)}]\!] = -\overline{\underline{\delta}}^{t(0)}[\![\underline{w}^{(0)}]\!]\tilde{\tilde{\delta}}_z^{t(0)}. \tag{43}$$

The kinematic boundary conditions (26) becomes

$$\frac{\partial \tilde{\tilde{\delta}}^{t(0)}}{\partial \tau} - \tilde{u}^{c(1)} + \underline{w}^{(0)}\tilde{\tilde{\delta}}_z^{t(0)} = 0, \tag{44}$$

where all velocity components are taken on the free boundary $\overline{r} = \overline{\underline{\delta}}^{t(0)\pm}$.

C. Dimensionless form of the linear equations

I define the following similarity functions for the perturbation

$$y^* = \eta^2 = y/\overline{\delta}^2,$$

$$\tau^* = \Gamma\tau/\overline{\delta}_0^2,$$

$$z^* = z\overline{\delta}/\overline{\delta}_0^2,$$

$$\tilde{\mathcal{K}}^{*(0)} = \tilde{\mathcal{K}}^{(0)}/\Gamma,$$

$$\tilde{\psi}^{*c(1)} = \tilde{\psi}^{c(1)}/(\Gamma\overline{\delta}),$$

$$\tilde{v}^{*(0)} = \tilde{v}^{(0)}\overline{\delta}/\Gamma,$$

$$\tilde{w}^{*(0)} = \tilde{w}^{(0)}\overline{\delta}/\Gamma,$$

$$\tilde{X}^{*b(1)} = \tilde{\mathbf{X}}^{(1)}\cdot\mathbf{b}(s,t)/(\overline{\delta}_0^2 K^{(0)}(s,t)).$$

For a vortex with the vorticity inside a vortex tube I also define

$$\tilde{\tilde{\delta}}^{*t(0)} = \tilde{\tilde{\delta}}^{t(0)}/\overline{\underline{\delta}}^{t(0)},$$

$$\tilde{p}^{*(0)} = (\overline{\underline{\delta}}^{t(0)})^2\tilde{p}^{(0)}/\Gamma^2.$$

These similarity functions are now used to simplify the system of linear equations for the small axial variations.

1. Continuous vorticity field

With these functions the system (39)–(41) becomes

$$\frac{\partial \tilde{\mathcal{K}}^{*(0)}}{\partial \tau^*} - 2\underline{\mathcal{K}}_{y^*}^{*(0)}\frac{\partial \tilde{\psi}^{*c(1)}}{\partial z^*} + 2R\underline{\psi}_{y^*}^{*c(1)}\frac{\partial \tilde{\mathcal{K}}^{*(0)}}{\partial z^*} = 0, \tag{45}$$

$$D^{*2}\frac{\partial \tilde{\psi}^{*c(1)}}{\partial \tau^*} + \frac{2}{y^*}\underline{\mathcal{K}}^{*(0)}\frac{\partial \tilde{\mathcal{K}}^{*(0)}}{\partial z^*} + R\mathcal{G}^*\left(\frac{\partial \tilde{\psi}^{*c(1)}}{\partial z^*}\right) = 0, \tag{46}$$

$$\frac{\partial \tilde{X}^{*b(1)}}{\partial \tau^*} = \frac{1}{4\pi}(\Delta C_v + \Delta C_w), \tag{47}$$

where

$$\mathcal{G}^* = 2\underline{\psi}_{y^*}^{*c(1)}D^{*2} - 8y^*\underline{\psi}_{y^*y^*y^*}^{*c(1)},$$

$$R = \frac{\overline{\delta}_0}{\overline{\delta}}\left(\frac{S_0^{(0)}}{S^{(0)}(t)}\right)^2,$$

$$\Delta C_v = 4\pi^2 \int_0^\infty \underline{\mathcal{K}}^{*(0)} \frac{\widetilde{\mathcal{K}}^{*(0)}}{y^*} dy^*,$$

$$\Delta C_w = 16\pi^2 \int_0^\infty R \underline{w}^{*(0)} \frac{\partial \widetilde{\psi}^{*c(1)}}{\partial y^*} dy^*,$$

and $D^{*2} = 4y^* \partial_{y^* y^*}$.

2. Discontinuous vorticity field

For a vortex with the vorticity inside a vortex tube, Eqs. (42)–(44) become

$$[[\widetilde{p}^{*(0)}]] = -\widetilde{\delta}^{*t(0)} [[\underline{v}^{*(0)^2}]], \tag{48}$$

$$\left[\left[\frac{\partial \widetilde{\psi}^{*c(1)}}{\partial z^*}\right]\right] = -R[[\underline{w}^{*(0)}]] \frac{\partial \tilde{\tilde{\delta}}^{*t(0)}}{\partial z^*}, \tag{49}$$

$$\frac{\partial \tilde{\tilde{\delta}}^{*t(0)}}{\partial \tau^*} + \frac{\partial \widetilde{\psi}^{*c(1)}}{\partial z^*} + R \underline{w}^{*(0)} \frac{\partial \tilde{\tilde{\delta}}^{*t(0)}}{\partial z^*} = 0. \tag{50}$$

Equations (13) and (16) become

$$\widetilde{p}^{*(0)} = -\int_{\sqrt{y^*}}^\infty \frac{\underline{\mathcal{K}}^{*(0)} \widetilde{\mathcal{K}}^{*(0)}}{y^{*2}} dy^*, \tag{51}$$

$$2\frac{\partial \widetilde{\psi}^{*c(1)}_{y^*}}{\partial \tau} + 4R\left(-\underline{\psi}^{*c(1)}_{y^* y^*} \frac{\partial \widetilde{\psi}^{*c(1)}}{\partial z^*} + \underline{\psi}^{*c(1)}_{y^*} \frac{\partial^2 \widetilde{\psi}^{*c(1)}}{\partial y^* \partial z^*} \right) + \frac{\partial \widetilde{p}^{*(0)}}{\partial z^*} = 0. \tag{52}$$

As the axial flux satisfies

$$m(t) = \left(\frac{S_0^{(0)}}{S^{(0)}(t)} \right)^2 m_0,$$

the swirl number satisfies $S_w(t) = R S_w(0)$, which means that R is the ratio $S_w(t)/S_w(0)$ if the axial flux is not zero: $S_w(0) \neq 0$.

If the base flow is a Rankine vortex core with a uniform axial jet or a similar vortex, I divide both the axial velocity $\underline{w}^{*(0)}$ and $\underline{\psi}^{*c(1)}$ by the initial swirl number $S_w(0)$: the base flow is then independent of any parameter and in Eqs. (45)–(47) R becomes the swirl number $S_w(t)$ at t. For these vortices the stability analysis only depends on this one-parameter $S_w(t)$ and I carry out this study in the following.

D. Eigenvalue equations for linear Fourier modes

I look for solution of the linear equations (45)–(47) in the form

$$\widetilde{\psi}^{*c(1)} = \hat{\Psi}(y^*, S_w(t)) e^{-i\omega_n^* \tau^* + in\hat{z}}, \tag{53}$$

$$\widetilde{\mathcal{K}}^{*(0)} = \hat{\mathcal{K}}(y^*, S_w(t)) e^{-i\omega_n^* \tau^* + in\hat{z}}, \tag{54}$$

$$\widetilde{X}^{*b(1)} = i\hat{X}^b(S_w(t)) e^{-i\omega_n^* \tau^* + in\hat{z}}, \tag{55}$$

where $\omega_n^* = \omega_n \overline{\delta_0^2}/\Gamma$ and $\hat{z} = 2\pi z^* \overline{\delta_0^2}/(S^{(0)}\overline{\delta})$. For a vortex with the vorticity inside a vortex tube I also look for the perturbation of the interface function and of the pressure in the form

$$\tilde{\tilde{\delta}}^{*t(0)} = \hat{\hat{\delta}}^t(S_w(t)) e^{-i\omega_n^* \tau^* + in\hat{z}},$$

$$\widetilde{p}^{*(0)} = \hat{p}(y^*, S_w(t)) e^{-i\omega_n^* \tau^* + in\hat{z}}.$$

1. Continuous vorticity field

The linear system satisfied by the eigenfunctions is

$$\hat{\mathcal{K}} = -2\lambda \underline{\mathcal{K}}^{*(0)}_{y^*} \hat{\psi}/g, \tag{56}$$

$$\frac{d^2\hat{\psi}}{dy^{*2}} - \frac{\lambda}{2gy^{*2}} \underline{\mathcal{K}}^{*(0)} \hat{\mathcal{K}} + \frac{2\lambda S_w}{g} \underline{\psi}^{*c(1)}_{y^* y^* y^*} \hat{\psi} = 0, \tag{57}$$

$$\hat{X}^b = \frac{\pi}{\omega_n^*}\left(\int_0^\infty \underline{\mathcal{K}}^{*(0)} \frac{\hat{\mathcal{K}}}{y^*} dy^* + \int_0^\infty 4 S_w \underline{w}^{*(0)} \frac{d\hat{\psi}}{dy^*} dy^* \right), \tag{58}$$

where $\lambda = 2\pi n \overline{\delta_0^2}/(S^{(0)}\overline{\delta}\omega_n^*)$ and

$$g = 1 - 2\lambda S_w \underline{\psi}^{*c(1)}_{y^*}.$$

The substitution of $\hat{\mathcal{K}}$ from Eq. (56) into Eq. (57) gives the following eigenvalue problem for $\hat{\psi}$ and λ:

$$\frac{d^2\hat{\psi}}{dy^{*2}} + G(y^*, \lambda, S_w)\hat{\psi} = 0, \tag{59}$$

$$\frac{d\hat{\psi}}{dy^*}(y^* \to \infty) = 0, \tag{60}$$

$$\hat{\psi}(y^* = 0) = 0, \tag{61}$$

where

$$G(y^*, \lambda, S_w) = \lambda^2 \frac{\underline{\mathcal{K}}^{*(0)} \underline{\mathcal{K}}^{*(0)}_{y^*}}{g^2 y^{*2}} + 2\lambda S_w \frac{\underline{\psi}^{*c(1)}_{y^* y^* y^*}}{g}.$$

From Eq. (59) it comes $\hat{\psi} = y^* + O(y^{*2})$ near 0, where I used a normalization condition to select any eigenfunction of this homogeneous equation.

2. Discontinuous vorticity field

For a vortex with the vorticity inside a vortex tube, the linear system must be completed by

$$[[\hat{p}]] = -\hat{\hat{\delta}}^t [[\underline{v}^{*(0)^2}]], \tag{62}$$

$$[[\hat{\psi}]] = -S_w[[\underline{w}^{*(0)}]]\hat{\hat{\delta}}^t, \tag{63}$$

$$\hat{p} = -\int_{\sqrt{y^*}}^\infty \frac{\underline{\mathcal{K}}^{*(0)} \hat{\mathcal{K}}}{y^{*2}} dy^*, \tag{64}$$

$$\hat{p} = 4S_w\left(\underline{\psi}^{*c(1)}_{y^* y^*} \hat{\psi} - \underline{\psi}^{*c(1)}_{y^*} \frac{d\hat{\psi}}{dy^*} \right) - \frac{2}{\lambda}\frac{d\hat{\psi}}{dy^*}, \tag{65}$$

and by the kinematic boundary condition

$$\hat{\hat{\delta}}^t = \lambda \hat{\psi}(1, S_w)/g, \tag{66}$$

where all velocity components are taken on the interface $\overline{r} = \underline{\overline{\delta}}^{t(0)\pm}$.

For a Rankine vortex core with a uniform axial jet, G is given by

$$G=\begin{cases}\Lambda^2/y^* & \text{if } y^*<1,\\ 0 & \text{if } y^*>1,\end{cases}$$

where

$$\Lambda^2=\frac{\lambda^2}{4\pi^2(1-\lambda S_w/\pi)^2},$$

and for a similar vortex, G is given by

$$G=\frac{\lambda^2}{4\pi^2}\frac{(1-e^{-y^*})e^{-y^*}}{y^{*2}g^2}+\frac{\lambda S_w}{\pi}\frac{e^{-y^*}}{g},$$

where $g=1-\lambda S_w e^{-y^*}/\pi$.

E. Examples of solutions

In this subsection I solve the eigenvalue equations for a Rankine and a similar vortex with a uniform axial jet.

1. Rankine vortex core with a uniform axial jet

For a Rankine vortex core with a uniform axial jet, the solution of the linear system (56)–(61) is

$$\frac{\overline{\delta S}^{(0)}\omega_n^*}{2n\overline{\delta}_0^2}=\frac{1}{\lambda/\pi}=S_w\pm\frac{1}{j_{0i}},$$

$$\hat{\delta}^t=2\pi J_1(2\Lambda),$$

$$\hat{\psi}=\begin{cases}\dfrac{\eta J_1(2\Lambda\eta)}{\Lambda} & \text{if } \eta<1,\\ J_1(2\Lambda)/\Lambda & \text{if } \eta>1,\end{cases}$$

$$\frac{d\hat{\psi}}{dy^*}=\begin{cases}J_0(2\Lambda\eta) & \text{if } \eta<1,\\ 0 & \text{if } \eta>1,\end{cases}$$

$$\hat{\mathcal{K}}=\begin{cases}-2\eta J_1(2\Lambda\eta) & \text{if } \eta<1,\\ 0 & \text{if } \eta>1,\end{cases}$$

$$\hat{p}=\begin{cases}-\dfrac{J_0(2\Lambda\eta)}{\Lambda\pi} & \text{if } \eta<1,\\ 0 & \text{if } \eta>1,\end{cases}$$

$$\hat{X}^b=-\frac{1}{\Lambda\omega_n^*}(J_2(2\Lambda)-4S_wJ_1(2\Lambda)),$$

where J_0, J_1 are Bessel functions of the first kind, and j_{0i} is the ith zero of the Bessel function J_0. The frequency ω_n of these oscillations is

$$\omega_n=n(\omega^{Sw}\pm\omega_i^0), \tag{67}$$

where

$$\omega^{Sw}=2\Gamma S_w/(\overline{\delta S}^{(0)}),$$

$$\omega_i^0=2\Gamma/(\overline{\delta S}^{(0)}j_{0i}).$$

Without axial flux $[S_w(t)=0]$ the shape of the filament and of the axial variations of the core are given by

$$\mathbf{X}^{(0)}=\underline{\mathbf{X}}^{(0)}+\varepsilon 2\mu(\overline{\delta}_0)^2\hat{X}^b(0)\cos(n\hat{z})\sin(n\omega_i^0\tau)\times K^{(0)}(s,t)\mathbf{b}(s,t), \tag{68}$$

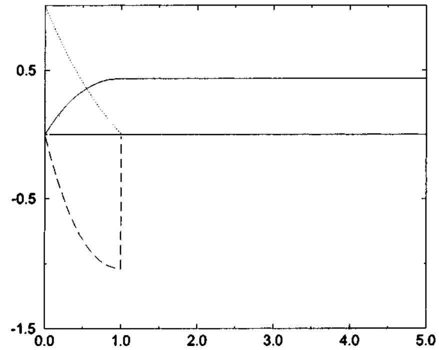

FIG. 1. The first eigenmode $\lambda_1/\pi=2.39$ of a perturbed Rankine vortex core without swirl $(S_w=0)$. The solid line is $\hat{\psi}(y^*,0)$, the dotted line is $d\hat{\psi}/dy^*$, and the dashed line is $\hat{\mathcal{K}}(y^*,0)$.

$$\overline{\delta}^{t(0)}=\underline{\overline{\delta}}^{t(0)}+2\mu\overline{\delta}_0\hat{\hat{\delta}}^t(0)\cos(n\hat{z})\cos(n\omega_i^0\tau), \tag{69}$$

with the associated velocity field

$$\mathcal{K}^{(0)}=\underline{\mathcal{K}}+2\mu\hat{\mathcal{K}}(y^*,0)\cos(n\hat{z})\cos(n\omega_i^0\tau), \tag{70}$$

$$\psi^{c(1)}=2\mu\hat{\psi}(y^*,0)\sin(n\hat{z})\sin(n\omega_i^0\tau). \tag{71}$$

The selected first eigenmode of the velocity field is given in Fig. 1 and the associated core-thickness and centerline evolutions are given in Fig. 2.

This result generalizes, to a vortex filament with a centerline of any shape, the bulging modes found by Kopiev and Chernyshev[9] on a perturbed vortex of circular centerline. In the peculiar case of a perturbed vortex ring with a circular centerline, the modes found in this theory are the same as in their theory. For example, the interface disturbance of these bulging modes was given in Kopiev and Chernyshev[9] in the absolute frame with help of the *displacement field* representation and one can show that it corresponds to the same

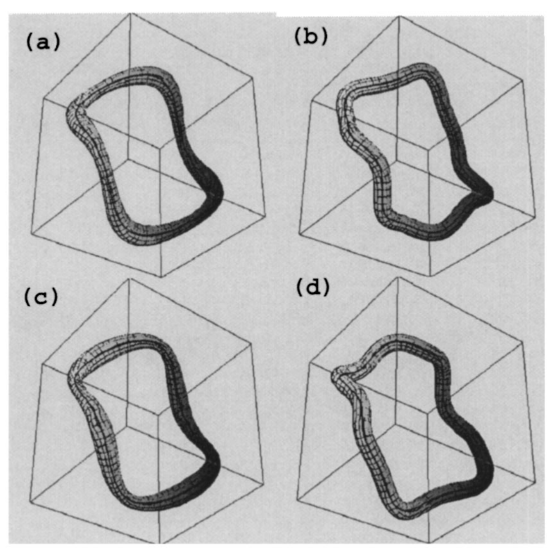

FIG. 2. Fast axial-core oscillations (mode $n=4$) of a vortex filament with a perturbed Rankine vortex core without swirl and the induced filament oscillations. (a) Initial variation of the core without perturbation of the filament, evolution on (b) one-fourth of the period, (c) half of the period, (d) three-fourths of the period.

binormal perturbation of the centerline and the same thickness perturbation as I found. In my approach it is easy to understand that the fast-oscillations of the centerline are in the binormal direction as the core of the vortex is known to be taken into account by core-functions coefficients C_v and C_w in the binormal term of the equation of motion of Callegari and Ting.[1] It is also easy to see the consistence of my generalization with the theory of Callegari and Ting[1] and of most studies of vortex dynamics as I use the same coordinates as most authors and I give the velocity field. In this sense this approach also completes the one of Kopiev and Chernyshev,[9] who introduced their own interesting coordinates and give the components of the displacement field and some components of the velocity field on their special coordinates. The frequency of the bulging modes are found to be the same as the one they found for the perturbed vortex ring with a circular centerline, provided that $\Gamma/(\overline{\delta}S^{(0)})$ is used to have a dimensionless frequency. My approach is restricted to axisymmetric perturbations: the bulging modes on the short-time τ and the bending modes on the normal-time t. Kopiev and Chernyshev[9] considered all nonshort wavelength perturbations and so also have the ultra-short-time $\overline{t}$ dynamics of nonaxisymmetric perturbations on the vortex ring with a circular centerline.

This study also generalizes, to a vortex filament with a centerline of any shape, the bulging modes found on a perturbed straight filament[28,30] in the long-wavelength limit. In the peculiar case of this straight filament the result is consistent with their long-wavelength limit for the axisymmetric and bending modes.

2. Similar vortex

For a similar vortex, Eq. (59) gives $\hat{\psi}=c_0+c_1y^*+O[\exp(-y^*)]$ at infinity, where c_0 and c_1 are two constants. For any value of λ, the solution of Eq. (59) and $\hat{\psi}\sim y^*$ near 0 asymptotically reaches a constant $\hat{\psi}_{y^*}(y^*\rightarrow\infty)$, that is zero only for an infinity number of selected values λ_i of λ. I use a shooting method and a Runge–Kutta solver to find these eigenvalues $\lambda_i(S_w)$. The frequency ω_n of these oscillations is

$$\omega_n=\frac{2\Gamma n}{\overline{\delta}S^{(0)}}\frac{1}{\lambda_i(S_w)/\pi}. \tag{72}$$

Without axial flux ($S_w=0$,) it gives $\lambda_1/\pi=\pm3.1$, $\lambda_2/\pi=\pm6.0$, and $\lambda_3/\pi=\pm9.1$. The first three selected eigenmodes are given in Figs. 3, 4, and 5.

With $S_w=0.1$, it gives $\lambda_1/\pi=(-3.3569,2.7520)$ and $\lambda_2/\pi=(-8.4073,4.4456)$. The selected first eigenmode $\lambda_1/\pi=2.7520$ is given in Fig. 6.

VI. ONE-TIME VORTEX RING BUBBLE

In this section I give the one-time equations of a vortex ring bubble without axial variations. A special transformation is then introduced to solve the core equations and the solutions to these equations are given. This gives coupled equations [Eqs. (79)–(80)] for the motion of the centerline of the vortex bubble and for the bubble thickness dynamics. These equations are used to study a circular vortex ring bubble and to compute the motion of a vortex ring bubble of elliptical shape. Finally the ultra-fast oscillations of a vortex ring bubble are studied on the ultra-short time scale.

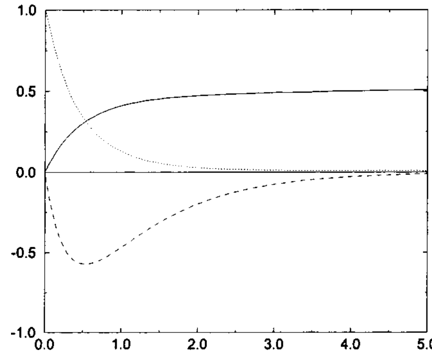

FIG. 3. The first eigenmode of a similar vortex without swirl ($S_w=0$). The solid line is $\hat{\psi}(y^*,0)$, the dotted line is $d\hat{\psi}/dy^*$, and the dashed line is $\hat{\mathcal{K}}(y^*,0)$. The associated filament eigenmode is $\hat{X}^b=-0.096$.

A. The one-time equations

The leading-order velocity fields without axial variations and $u^{c(1)}=0$ (and thus $D^{(1)}=0$) are also solutions of the leading-order compatibility conditions (13)–(16) and (27)–(28) in the case of a vortex bubble. The non-axial variations of the thickness $\overline{\delta}^{b(0)}(s,t)=\overline{\delta}^{b(0)}(t)$ of the bubble gives the simplification $\mathcal{V}^{b(0)}=S^{(0)}\,(\overline{\delta}^{b(0)})^2$ in Eq. (11).

For a closed vortex ring bubble, the s-average of the axisymmetric part of the continuity equation at second order [Eq. (B2) in Appendix B] gives

$$\langle\sigma^{(0)}u^{c(2)}\rangle=-\frac{\dot{S}^{(0)}\overline{r}}{2}+\frac{D^{c(2)}(t)S^{(0)}}{\overline{r}},$$

where $\langle\ \rangle$ denotes the s-average. Here, $D^{c(2)}(t)\neq0$ is allowed because a singularity can exist at $\overline{r}=0$ and is required to satisfy the s-average of Eq. (10):

$$D^{c(2)}(t)=\overline{\delta}^{b(0)}\left(\frac{\partial\overline{\delta}^{b(0)}}{\partial t}+\frac{1}{2}\frac{\dot{S}^{(0)}}{S^{(0)}}\overline{\delta}^{b(0)}\right), \tag{73}$$

where $\overline{\delta}^{b(0)}$ is given by Eq. (27).

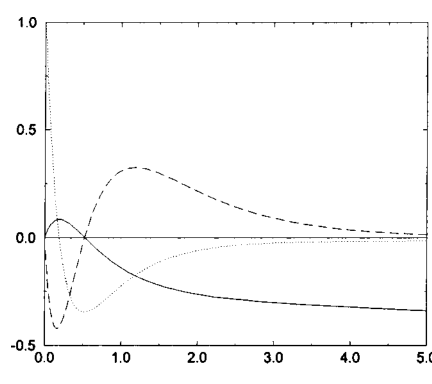

FIG. 4. The second eigenmode of a similar vortex without swirl ($S_w=0$). The solid line is $\hat{\psi}(y^*,0)$, the dotted line is $d\hat{\psi}/dy^*$, and the dashed line is $\hat{\mathcal{K}}(y^*,0)$.

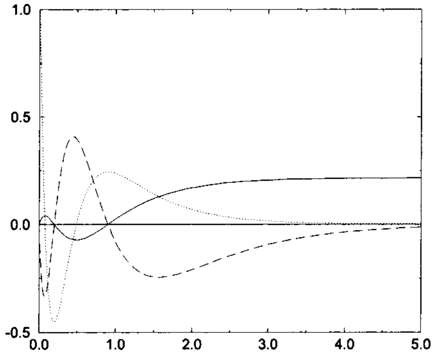

FIG. 5. The third eigenmode of a similar vortex without swirl ($S_w = 0$). The solid line is $\hat{\psi}(y^*)$, the dotted line is $d\hat{\psi}/dy^*$, and the dashed line is $\hat{\mathcal{K}}(y^*)$.

The s-average of quasi-stationary (one-time) solutions of axisymmetric equations at second order [Eqs. (B1)–(B4) in Appendix B] satisfies

$$\frac{\partial v^{(0)}}{\partial t} - \overline{\nu}\zeta^{(0)}_{\overline{r}} = \frac{1}{2}\overline{r}\zeta^{(0)}\left[\frac{\dot{S}^{(0)}}{S^{(0)}} - \frac{2D^{c(2)}(t)}{\overline{r}^2}\right], \qquad (74)$$

$$\frac{\partial w^{(0)}}{\partial t} - \overline{\nu}\frac{1}{\overline{r}}[\overline{r}w^{(0)}_{\overline{r}}]_{\overline{r}} = \frac{1}{2}\overline{r}^3\left(\frac{w^{(0)}}{\overline{r}^2}\right)_{\overline{r}}\frac{\dot{S}^{(0)}}{S^{(0)}} - \overline{r}w^{(0)}_{\overline{r}}\frac{D^{c(2)}(t)}{\overline{r}^2}, \qquad (75)$$

where the leading-order quasi-stationary velocity is without axial variations and $u^{c(1)} = 0$ as previously stated. Equations (31), (27), (73), (74), and (75) are a complete set of equations for the one-time solution.

B. Transformation of the equations

I now solve Eqs. (73)–(75). I use the following transformation first introduced by Callegari and Ting[1] (often referred as the transformation of Lundgren[32]):

$$t_1 = \int_0^t S^{(0)}(t')dt',$$

$$\xi = \overline{r}\sqrt{S^{(0)}(t)},$$

$$W(\xi,t_1) = S^{(0)}(t)w^{(0)},$$

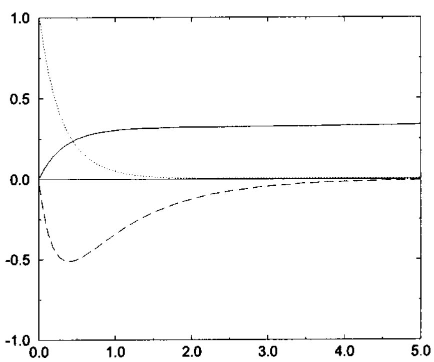

FIG. 6. The first eigenmode of a similar vortex with swirl $S_w = 0.1$. The solid line is $\hat{\psi}(y^*,0.1)$, the dotted line is $d\hat{\psi}/dy^*$, and the dashed line is $\hat{\mathcal{K}}(y^*,0.1)$.

$$Z(\xi,t_1) = \zeta^{(0)}/S^{(0)}(t).$$

The previous equations become

$$2D^{c(2)}(t_1) = \frac{\partial[(\overline{\delta}^{b(0)})^2S^{(0)}]}{\partial t_1} = \frac{\partial\mathcal{V}^{\,b(0)}}{\partial t_1}, \qquad (76)$$

$$\frac{\partial\chi}{\partial t_1} - \overline{\nu}\frac{1}{\xi}(\xi\chi_\xi)_\xi = -D^{c(2)}(t)\frac{\chi_\xi}{\xi}, \qquad (77)$$

where χ stands for W and Z. I then use the following transformation

$$\xi_1 = \xi^2 - (\overline{\delta}^{b(0)})^2S^{(0)} = [\overline{r}^2 - (\overline{\delta}^{b(0)})^2]S^{(0)},$$

which yields

$$\frac{\partial\chi}{\partial t_1} = 4\overline{\nu}([\xi_1 + (\overline{\delta}^{b(0)})^2S^{(0)}]\chi_{\xi_1})_{\xi_1}. \qquad (78)$$

The bubble allows to have a solution of the equation $\zeta^{(0)} = (\overline{r}v^{(0)})_{\overline{r}}/\overline{r}$ in the form

$$v^{(0)} = \frac{\Gamma_1}{2\pi\overline{r}} + v^{\,\omega(0)},$$

where $v^{\,\omega(0)}$ is regular at $\overline{r} = 0$, and to have the associated circulation field

$$\mathcal{K}^{(0)} = \frac{\Gamma_1}{2\pi} + \mathcal{K}^{\omega(0)}.$$

C. The one-time solutions in dimensionless form

I define the following similarity functions

$$v^{*(0)} = v^{(0)}\overline{\delta}/\Gamma,$$

$$v^{*\omega(0)} = v^{\,\omega(0)}\overline{\delta}/\Gamma,$$

$$\zeta^{*(0)} = \zeta^{(0)}\frac{\overline{\delta}^2}{\Gamma}\left(\frac{\mathcal{V}}{\mathcal{V}_0}\right)^{-1},$$

$$w^{*(0)} = w^{(0)}\frac{\overline{\delta}^2}{\Gamma\overline{\delta}_0}\left(\frac{\mathcal{V}}{\mathcal{V}_0}\right)^{-1}\left(\frac{S_0^{(0)}}{S^{(0)}(t)}\right)^{-2}(1-\mathcal{V}_0^{\,b}/\mathcal{V}_0)^{1/2},$$

$$\mathcal{K}^{*(0)} = \mathcal{K}^{(0)}/\Gamma,$$

$$\mathcal{K}^{*\omega(0)} = \mathcal{K}^{\omega(0)}/\Gamma,$$

$$\psi^{*c(1)} = \psi^{c(1)}\frac{1}{\Gamma\overline{\delta}_0}\left(\frac{S_0^{(0)}}{S^{(0)}(t)}\right)^{-2}(1-\mathcal{V}_0^{\,b}/\mathcal{V}_0)^{1/2},$$

where $\overline{\delta}_0$, $\mathcal{V}_0 \equiv (\overline{\delta}_0)^2S_0^{(0)}$, and $S_0^{(0)}$ are the initial thickness, volume, and length of the vortex and $\overline{\delta}$, $\mathcal{V} \equiv (\overline{\delta})^2S^{(0)}$, and $S^{(0)}$ their values at time t.

If the fluid is inviscid ($\overline{\nu} = 0$), the solutions are in the form

$$\zeta^{*(0)} = \zeta_0^*(y),$$

$$\mathcal{K}^{*(0)} = \mathcal{K}_0^*(y) = \frac{1}{2\pi}\frac{\Gamma_1}{\Gamma} + \frac{1}{2}\int_0^y \zeta_0^*(y')dy',$$

$$v^{*(0)}=\frac{\mathcal{K}_0^*(y)}{\overline{r}/\overline{\delta}},$$

$$w^{*(0)}=w_0^*(y),$$

where

$$y=\left(\frac{\overline{r}}{\overline{\delta}}\right)^2(1+\beta(t))-\beta(t),$$

with $\beta(t)=\mathcal{V}^{b(0)}(t)/(\mathcal{V}_0-\mathcal{V}_0^{b(0)})$ and $\overline{\delta}(t)$ is the ε-stretched thickness of the core

$$\mathcal{V}\equiv(\overline{\delta})^2S^{(0)}=\mathcal{V}_0+\mathcal{V}^{b(0)}-\mathcal{V}_0^{b(0)}.$$

Here, $(\mathcal{K}_0^*(y),w_0^*(y))$ are the initial circulation and axial velocity fields.

The equation of the filament motion (31) is

$$\partial_t\mathbf{X}^{(0)}(x,t)=\mathbf{A}(s,t)+\Gamma\frac{K^{(0)}(s,t)}{4\pi}[\log(S^{(0)}/\varepsilon)-1+C_v(t)+C_w(t)+\overline{W}_e(t)]\mathbf{b}(s,t), \tag{79}$$

where $\beta(t)$ [and so $\mathcal{V}^{b(0)}(t)$ or $\overline{\delta}^{b(0)}$] is obtained from Eq. (27) and is solution of

$$\tilde{P}_v\left(\frac{\beta(t)}{\beta(0)}\right)-\tilde{P}_{g0}\left(\frac{\beta(t)}{\beta(0)}\right)^{(1-k)}+\frac{S^{(0)}}{S_0^{(0)}}(C_p[\beta(t)]+\overline{W}_e[\beta(t)])=0, \tag{80}$$

with

$$\tilde{P}_v=4\pi^2(\overline{\delta}_0^{b(0)})^2(p^{(0)}(\infty)-\overline{P}_v)/\Gamma^2,$$

$$\tilde{P}_{g0}=4\pi^2(\overline{\delta}_0^{b(0)})^2-\overline{P}_{g0}/\Gamma^2.$$

These coupled equations (79)–(80) for the motion of the centerline of the vortex bubble and for the bubble thickness dynamics generalize the equation of motion of Genoux[14] to a non-potential vortex bubble with a filament of any shape and with axisymmetric time-variations of its thickness. The expressions of the core functions $C_v(t)$ and $C_w(t)$, C_p and $\overline{W}_e$ that appear in these equations are given in the following.

1. Inviscid fluid

The associated core functions are

$$C_v(t)=\frac{1}{2}+\lim_{y\to+\infty}\left(2\pi^2\int_0^y\frac{\mathcal{K}_0^{*2}(y')}{y'+\beta(t)}dy'-\frac{1}{2}\log(y+\beta(t))\right)-\frac{1}{2}\log\left(\frac{\mathcal{V}}{S^{(0)}}\right)+\frac{1}{2}\log(1+\beta(t)),$$

$$C_w(t)=-4\pi^2(S_0^{(0)}/S^{(0)}(t))^3\int_0^\infty w_0^{*2}(y)dy,$$

$$C_p(t)=-2\pi^2\beta(t)\int_0^\infty\frac{\mathcal{K}_0^{*2}(y)}{(y+\beta(t))^2}dy,$$

$$\overline{W}_e(t)=\overline{W}_e(0)\sqrt{\frac{S_0^{(0)}}{S^{(0)}}}\sqrt{\frac{\beta(t)}{\beta(0)}},$$

where $\overline{W}_e(0)=4\pi^2\overline{\delta}_0^{b(0)}\overline{\Upsilon}/\Gamma^2$.

2. Discontinuous vorticity field

For a vortex bubble with the vorticity inside a vortex tube, the s-average of quasi-stationary (one-time) solutions of the first-order interface equation [Eq. (B6) in Appendix B] satisfies

$$\frac{\partial\overline{\delta}^{t(0)}}{\partial t}+\frac{1}{2}\frac{\dot{S}^{(0)}}{S^{(0)}}\overline{\delta}^{t(0)}=\frac{D^{c(2)}(t)}{\overline{\delta}^{t(0)}}, \tag{81}$$

where I used the value of $\langle\sigma^{(0)}u^{c(2)}\rangle$. From this equation and Eq. (73), it comes

$$\mathcal{V}^{t(0)}=\mathcal{V}_0^{t(0)}+\mathcal{V}^{b(0)}-\mathcal{V}_0^{b(0)}, \tag{82}$$

where $\mathcal{V}^{t(0)}=(\overline{\delta}^{t(0)})^2\,S^{(0)}(t)$ is the volume of the vortex tube. This equation is coherent with the volume

$$\mathcal{V}=\mathcal{V}_0+\mathcal{V}^{b(0)}-\mathcal{V}_0^{b(0)}$$

previously introduced to describe the volume of the continuous vorticity field in an inviscid fluid ($\overline{\nu}=0$). In this case, I choose the thickness $\overline{\delta}$ of the vortex to be $\overline{\delta}^t$.

For a vortex bubble with a core of Rankine type and a uniform axial jet:

$$\mathcal{K}_0^*(y)=\begin{cases}\dfrac{1}{2\pi}\left(\dfrac{\Gamma_1}{\Gamma}(1-y)+y\right) & \text{if } 0<y<1,\\[2ex] \dfrac{1}{2\pi} & \text{if } y>1,\end{cases} \tag{83}$$

$$w_0^*(y)=\begin{cases}S_w(0)/\pi & \text{if } 0<y<1,\\ 0 & \text{if } y>1,\end{cases} \tag{84}$$

where $S_w(0)=m_0/(\Gamma\overline{\delta}_0\sqrt{1-\mathcal{V}_0^b/\mathcal{V}_0})$ is the initial swirl number, and m_0 is the initial axial flux. The swirl number $S_w(t)$ at time t is defined by $S_w(t)=m(t)/(\Gamma\overline{\delta}\sqrt{1-\mathcal{V}^b/\mathcal{V}})$, where $m(t)$ is the axial flux at time t. These two swirl numbers are related by

$$S_w(t)=\frac{\overline{\delta}_0}{\overline{\delta}}\left(\frac{S_0^{(0)}}{S^{(0)}(t)}\right)^2\left(\frac{\mathcal{V}}{\mathcal{V}_0}\right)^{1/2}S_w(0).$$

It comes

$$C_v(t)=-\frac{1}{2}\log\left(\frac{\mathcal{V}}{S^{(0)}}\right)+\frac{1}{2}\left(1+\int_0^1\frac{[(\Gamma_1/\Gamma)(1-y)+y]^2}{y+\beta(t)}dy\right),$$

$$C_w(t)=-4S_w^2(0)(S_0^{(0)}/S^{(0)}(t))^3,$$

$$C_p(t)=-\frac{1}{2}\beta(t)\left(\frac{1}{1+\beta(t)}+\int_0^1\frac{[(\Gamma_1/\Gamma)(1-y)+y]^2}{(y+\beta(t))^2}dy\right).$$

The relative velocity field of this Rankine vortex core with a uniform axial jet depends only on two parameters: the ratio Γ_1/Γ and the initial swirl number $S_w(0)$. It depends on only one parameter if the axial velocity $w^{*(0)}$ is divided by $S_w(0)$.

In the case of a potential vortex ring bubble ($\Gamma_2=0$), the added condition $\boldsymbol{\omega}=0$ implies the following leading-order velocity field: $v^{(0)}=\Gamma_1/(2\pi\overline{r})$ and $w^{(0)}=0$. It comes

$$C_v(t)=-\frac{1}{2}\log\left(\frac{\mathcal{V}}{S^{(0)}}\right)+\frac{1}{2}\left(1+\log\frac{1+\beta}{\beta}\right),$$

$$=\frac{1}{2}-\log\overline{\delta}^{b(0)},$$

$$C_w(t)=0,$$

$$C_p(t)=-\frac{1}{2}.$$

D. Study of typical cases

1. Isothermal transformation

In the peculiar interesting case $k=1$ (isothermal transformation[14]), Eq. (80) is the following polynomial of second degree in x:

$$x^2+ax-1=0,$$

where

$$x\equiv\sqrt{\beta(t)/\beta(0)}\Bigg/\sqrt{\tilde{P}_{g0}/\tilde{P}_v+\frac{1}{2}\frac{S^{(0)}}{S_0^{(0)}}\frac{1}{\tilde{P}_v}},$$

$$a\equiv\overline{W}_e(0)\Bigg/\left(\sqrt{\tilde{P}_v}\sqrt{\frac{1}{2}+\tilde{P}_{g0}\frac{S_0^{(0)}}{S^{(0)}}}\right).$$

The solution is

$$x=\frac{1}{2}(-a+\sqrt{4+a^2}). \tag{85}$$

In order to find the thickness of the bubble $\overline{\delta}^{b(0)}(t)$ at time t we compute a, deduce x from Eq. (85), obtain $\sqrt{\beta(t)/\beta(0)}$ from the definition of x and use $\sqrt{\beta(t)/\beta(0)}=\overline{\delta}^{b(0)}(t)/\overline{\delta}_0^{b(0)}$. This thickness exists for any values of the parameters and at any time. It decreases with increasing values of the surface tension parameter $\overline{W}_e$. Its initial value $\overline{\delta}_0^{b(0)}$ is found by solving

$$(p^{(0)}(\infty)-\overline{P}_v-\overline{P}_{g0})(\overline{\delta}_0^{b(0)})^2+\overline{\Upsilon}\,\overline{\delta}_0^{b(0)}=\frac{\Gamma^2}{8\pi^2}, \tag{86}$$

which is Eq. (80) at $t=0$.

In Fig. 7, I give the initial thickness $\varepsilon\overline{\delta}_0^{b(0)}$ as a function of the surface tension Υ. The physical parameters are $\Gamma=1\ \mathrm{m^2/s}$, $P_{g0}=0.2870(273+20)\ \mathrm{atm/m^3/kg}$, $p(\infty)=101.3\ \mathrm{atm/m^3/kg}$, $P_v=2.026\ \mathrm{atm/m^3/kg}$. Let us recall that this surface tension and these pressures are divided by the mass density $\rho=1000\ \mathrm{kg/m^3}$ and that $1\ \mathrm{atm}=1.013\times10^5$ Pa.

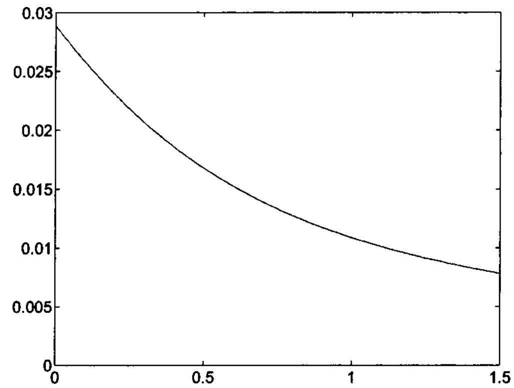

FIG. 7. Thickness $\varepsilon\overline{\delta}_0^{b(0)}$ (m) of a potential vortex ring bubble versus the surface tension Υ ($\mathrm{Nm^2/kg}$). The physical parameters are $\Gamma=1\ \mathrm{m^2/s}$, $P_{g0}=0.2870(273+20)\ \mathrm{atm/m^3/kg}$, $p(\infty)=101.3\ \mathrm{atm/m^3/kg}$, $P_v=2.026\ \mathrm{atm/m^3/kg}$.

In the case of a *circular* vortex the global integral $\mathbf{A}$ is $\mathbf{A}=\Gamma K\log(8/2\pi)\mathbf{b}/(4\pi)$ and the velocity of this bubble is

$$V=\frac{\Gamma}{4\pi}K\left[\log\frac{8}{\varepsilon\overline{\delta}^{b(0)}}-\frac{1}{2}+4\pi^2\overline{\Upsilon\,\delta}^{b(0)}/\Gamma^2\right].$$

Figure 8 shows the velocity V as a function of the surface tension Υ of a circular vortex ring bubble.

Figure 9 shows the evolution of a perturbed circular vortex bubble in the moving frame of the nonperturbed vortex. The perturbation is of elliptic shape (mode 2 of the polar Fourier expansion[11]) and its amplitude is 0.15. The computation was performed with the *EZ_vortex* code (see our submitted paper, Margerit *et al.*, "Implementation and validation of a slender vortex filament code: Its application to the study of a four-vortex wake model") by implementing the bubble thickness equations (85)–(86) and the Weber number $\overline{W}_e$ computation.

2. Almost adiabatical transformation

In the peculiar interesting case $k=1.5$ (close to the adiabatical transformation[14] $k=1.4$), Eq. (80) is the following polynom of degree three in x:

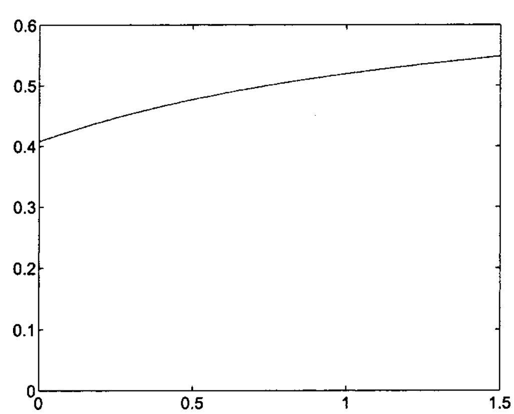

FIG. 8. Velocity V (m/s) of a circular vortex ring bubble versus the surface tension Υ ($\mathrm{Nm^2/kg}$). The physical parameters are as in Fig. 7.

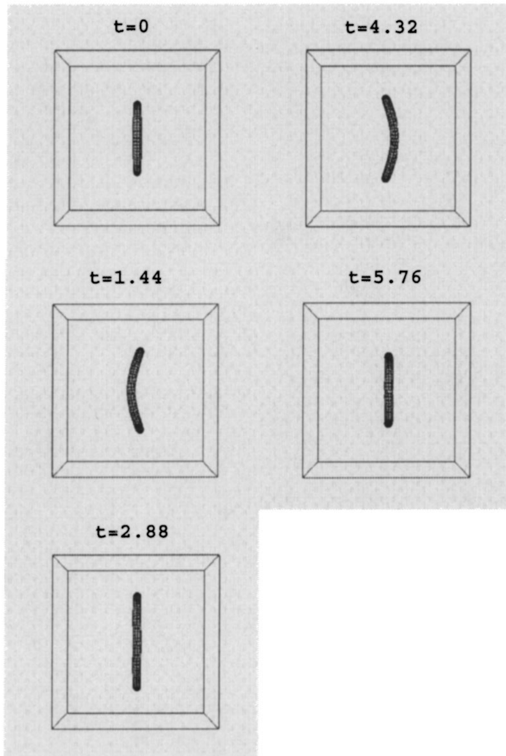

FIG. 9. Numerical simulation of the motion of a potential vortex ring bubble of elliptical shape (mode$=2$, amplitude$=0.15$) in the isothermal $k=1$ case. The physical parameters are as in Fig. 7 and $\Upsilon=0.7$. The frame is moving with the unperturbed circular vortex ring bubble velocity.

$$x^3+bx^2\sqrt{a}-\frac{a}{2}x-1=0,$$

where

$$x\equiv\sqrt{\beta(t)/\beta(0)}/(\tilde{P}_{g0}/\tilde{P}_v)^{1/3},$$

$$a\equiv(S^{(0)}/S_0^{(0)})(\tilde{P}_{g0}/\tilde{P}_v)^{-2/3}/\tilde{P}_v,$$

$$b\equiv\overline{W}_e/\sqrt{\tilde{P}_v}.$$

The solution is

$$x=\left(d+6\frac{a}{d}+4\frac{ab^2}{d}-2b\sqrt{a}\right)/6$$

with

$$d^3\equiv-18a^{3/2}b+108-8a^{3/2}b^3 +6\sqrt{3}\sqrt{-2a^3-a^3b^2-36a^{3/2}b+108-16a^{3/2}b}.$$

This shows that a thickness of the bubble $\overline{\delta}^{b(0)}(t)$ not always exists. The values allowed for a is between 0 and a maximum value, which decreases with increasing values of b and so of the surface tension parameter $\overline{W}_e$. This maximum value is almost $a=4$ when $b=0$. The thickness decreases with both increasing values of the surface tension parameter $\overline{W}_e$, and of a.

E. Ultra-fast oscillations of a vortex ring bubble

The potential vortex cannot support axisymmetric axial variations on a short-time τ, because the only potential fields that are solutions of the two-time-scale equations are $v^{(0)}=\Gamma_1/(2\pi\overline{r})$ and $w^{(0)}=0$, which are without axial variation. However, axial variation are possible on a ultra-short time $\overline{t}=t/\varepsilon^2$.

In this subsection I give the system of equations [Eqs. (92)–(95)] for the ultra-short-time dynamics of axial variations on the vortex ring bubble. I solve this system and give a closed equation [Eq. (97)] for the the ultra-short-time dynamics for the thickness of the bubble.

1. General equations

The bubble allows to have a solution of the leading-order equation of continuity $(\overline{r}u^{(0)})_{\overline{r}}=0$ in the form

$$u^{(0)}=\frac{\mathcal{D}^{(0)}(s,\overline{t},\tau,t)}{\overline{r}}.$$

At leading order the Navier–Stokes equations are

$$\frac{\partial u^{(0)}}{\partial\overline{t}}+p_{\overline{r}}^{(0)}-\frac{v^{(0)^2}}{\overline{r}}+u^{(0)}u_{\overline{r}}^{(0)}=0, \tag{87}$$

$$\frac{\partial v^{(0)}}{\partial\overline{t}}+\zeta^{(0)}u^{(0)}=0, \tag{88}$$

$$\frac{\partial w^{(0)}}{\partial\overline{t}}+w_{\overline{r}}^{(0)}u^{(0)}=0, \tag{89}$$

where $\zeta^{(0)}=(\overline{r}v^{(0)})_{\overline{r}}/\overline{r}$ is the leading-order axial vorticity. Equation (87) gives the pressure

$$p^{(0)}=-\int_{\overline{r}}^{\infty}\frac{v^{(0)^2}}{\overline{r}}d\overline{r}+p(\infty)-\frac{\mathcal{D}^{(0)^2}}{2\overline{r}^2}-\frac{\partial\mathcal{D}^{(0)}}{\partial\overline{t}}\log\overline{r}, \tag{90}$$

where the $\log\overline{r}$ term has to be matched with the outer velocity induced by a sink concentrated on the leading-order centerline $\mathbf{X}^{(0)}$.

The leading order of the axisymmetric part of the dynamical equation of the free-boundary (10) is

$$\frac{\partial\overline{\delta}^{b(0)}}{\partial\overline{t}}-u^{(0)}=0, \tag{91}$$

where all velocity components are taken on the free-boundary $\overline{r}=\overline{\delta}^{b(0)+}$. As the thickness $\overline{\delta}^{b(0)}$ is given by Eq. (27), this Eq. (91) is indeed the equation for $\mathcal{D}^{(0)}$ and gives

$$\mathcal{D}^{(0)}=\frac{1}{2}\frac{\partial(\overline{\delta}^{b(0)})^2}{\partial\overline{t}}. \tag{92}$$

The leading-order of the axisymmetric part of the pressure jump (9) is

$$p^{(0)}(\overline{\delta}^{b(0)+})=\left(\frac{\Gamma}{2\pi}\right)^2\frac{C_p(s,\overline{t},\tau,t)}{(\overline{\delta}^{b(0)})^2}+p(\infty)$$
$$-\frac{1}{2}\left[\left(\frac{\partial\overline{\delta}^{b(0)}}{\partial\overline{t}}\right)^2+\frac{\partial^2(\overline{\delta}^{b(0)})^2}{\partial\overline{t}^2}\log\overline{\delta}^{b(0)}\right]$$
$$=P_v+P_{g0}\left(\frac{\mathcal{V}_0^{b(0)}}{\mathcal{V}^{b(0)}}\right)^k-\frac{\overline{\Upsilon}}{\overline{\delta}^{b(0)}}, \tag{93}$$

where

$$C_p(s,\overline{t},\tau,t)=-\left(\frac{2\pi\overline{\delta}^{b(0)}}{\Gamma}\right)^2\int_{\overline{\delta}^{b(0)}}^{\infty}\frac{v^{(0)^2}(\overline{r},s,\overline{t},\tau,t)}{\overline{r}}d\overline{r}.$$

Here, we used the relation

$$-\frac{\mathcal{D}^{(0)^2}}{2(\overline{\delta}^{b(0)})^2}-\frac{\partial\mathcal{D}^{(0)}}{\partial\overline{t}}\log\overline{\delta}^{b(0)}$$
$$=-\frac{1}{2}\left[\left(\frac{\partial\overline{\delta}^{b(0)}}{\partial\overline{t}}\right)^2+\frac{\partial^2(\overline{\delta}^{b(0)})^2}{\partial\overline{t}^2}\log\overline{\delta}^{b(0)}\right],$$

which can easily be checked from Eq. (92).

Equation (93) is the equation of the thickness $\overline{\delta}^{b(0)}$ of the bubble. It is coupled with the ultra-short-time dynamics of the core, given by Eqs. (88)–(89), which can be written

$$\frac{\partial\zeta^{(0)}}{\partial\overline{t}}+\frac{\zeta_{\overline{r}}^{(0)}}{\overline{r}}\mathcal{D}^{(0)}=0, \tag{94}$$

$$\frac{\partial w^{(0)}}{\partial\overline{t}}+\frac{w_{\overline{r}}^{(0)}}{\overline{r}}\mathcal{D}^{(0)}=0. \tag{95}$$

The system of Eqs. (92)–(95) is a closed system for the ultra-short-time dynamics of axial variations on the vortex ring bubble.

2. Solution

I use the following transformation

$$\xi_1=\overline{r}^2-(\overline{\delta}^{b(0)})^2,$$

which yields

$$\frac{\partial\chi}{\partial\overline{t}}=0, \tag{96}$$

where χ stands for $\zeta^{(0)}$ and $w^{(0)}$. The solutions are in the form

$$\zeta^{(0)}=\zeta^{(0)}(\xi_1,s,\tau,t),$$

$$\mathcal{K}^{(0)}=\mathcal{K}^{(0)}(\xi_1,s,\tau,t)=\frac{\Gamma_1}{2\pi}+\frac{1}{2}\int_0^{\xi_1}\zeta^{(0)}(\xi_1',s,\tau,t)d\xi_1',$$

$$v^{(0)}=\frac{\mathcal{K}^{(0)}(\xi_1,s,\tau,t)}{\overline{r}},$$

$$w^{(0)}=w^{(0)}(\xi_1,s,\tau,t).$$

The dynamical equation of the bubble thickness $\overline{\delta}^{b(0)}(s,\overline{t},\tau,t)$ on this ultra-short-time is

$$\left(\frac{\partial\overline{\delta}^{b(0)}}{\partial\overline{t}}\right)^2+\frac{\partial^2(\overline{\delta}^{b(0)})^2}{\partial\overline{t}^2}\log\overline{\delta}^{b(0)}$$
$$=2(p(\infty)-P_v)-2P_{g0}\left(\frac{\mathcal{V}_0^{b(0)}}{\mathcal{V}^{b(0)}}\right)^k$$
$$+\frac{\Gamma^2}{2\pi^2}\frac{C_p(s,\overline{t},\tau,t)}{(\overline{\delta}^{b(0)})^2}+\frac{2\overline{\Upsilon}}{\overline{\delta}^{b(0)}}, \tag{97}$$

where

$$\frac{\mathcal{V}_0^{b(0)}}{\mathcal{V}^{b(0)}}=\frac{\int_0^{2\pi}\sigma_0^{(0)}(s)[\overline{\delta}_0^{b(0)}(s)]^2ds}{\int_0^{2\pi}\sigma^{(0)}(s,t)[\overline{\delta}^{b(0)}(s,\tau,\overline{t},t)]^2ds},$$

$$C_p(s,\overline{t},\tau,t)=-2\pi^2\left(\frac{\overline{\delta}^{b(0)}}{\Gamma}\right)^2\int_0^{\infty}\frac{\mathcal{K}^{(0)^2}(\xi_1,s,\tau,t)}{(\xi_1+(\overline{\delta}^{b(0)}))^2}d\xi_1,$$

and

$$\mathcal{K}^{(0)}=\frac{\Gamma_1}{2\pi}+\frac{1}{2}\int_0^{\xi_1}\zeta^{(0)}(\xi_1',s,\tau,t)d\xi_1'.$$

Here, the leading-order vorticity function $\zeta^{(0)}[\xi_1=\overline{r}^2-(\overline{\delta}^{b(0)})^2,s,\tau,t]$ is given initially and its short-time τ and normal-time t evolution are given by the equations at next orders. This equation (97) generalizes the one of Genoux[14] to a nonpotential vortex bubble with a filament of any shape and with axisymmetric axial variations of its thickness. The stationary solution of this equation (when it exists) is without axial variations and satisfies Eq. (80).

VII. CONCLUSION

This two-time-scale asymptotic approach allows us to derive, from the Navier–Stokes equations, the dynamics of the axial core-variations of axisymmetric shape on a vortex filament. This gives an extension of the one-time-scale asymptotic theory of Callegari and Ting[1] of vortex filament motion. This asymptotic theory is also an alternative to different ad hoc models of vortex filament with axial core-variations proposed by Marshall,[3,4] Leonard,[5] and Lundgren.[6] The dynamics of these axial variations is on a short-time scale and is inviscid at leading and first orders. These axial variations induce a small amplitude (first-order) motion of the curved centerline on the short-time scale. This motion is in the binormal direction of the leading-order centerline. The solutions of the two-time-scale equations have been given for axial core variations of small amplitude. More theoretical and numerical work is required to study the finite amplitude regime.

The theory of Genoux[14] of vortex ring bubbles has been extended to a vortex filament bubble with a centerline of any shape and with a nonpotential core.

The axisymmetric part of the velocity field at first order (it is the next order to the leading order) was proved to be composed of two parts: a part without axial variations and a part with axial variations. The expression of this second part was proved to be unique and to be related to the local stretching of the centerline. This form of the velocity field at first order was chosen by Margerit,[17] who gave the one-time dynamical equations satisfied by the part without axial variations of this first-order axisymmetric part of the velocity field: it is the generalization of the Callegari and Ting theory to the next order.

The implementation of the first-order thickness correction in a numerical code of slender vortex filament motion is currently under investigation. The associated first-order correction to the leading-order corrected vortex filament methods of Klein and Knio[2,33] for slender vortex filament is also under investigation. I hope to extend these computations to vortex filaments with thicker cores and to be able to perform quantitative comparisons between these numerical computations and direct numerical computation of the Navier–Stokes equations.

ACKNOWLEDGMENTS

I would like to thank Dr. Max O. Souza, Professor Stephen J. Cowley, and Dr. Pierre Brancher for their stimulating discussions. I also thank Lu. Ting for electronic correspondence at the beginning of this work: it was very encouraging for me and helped me to go ahead.

APPENDIX A: THE LEADING-ORDER SHORT-TIME ASYMMETRIC DYNAMICS IN THE FILAMENT

In this appendix I give the leading-order equations of the short-time asymmetric dynamics for axial variations in the filament. This dynamics is slaved by the axisymmetric one. These equations come from the asymmetric part of the equations at first order. They are used in Sec. III B to perform the matching between the outer and inner velocity fields and so to obtain the equation of motion of the centerline. A stream function is introduced and the whole asymmetric field is described with this function. The stream function solution is given for a continuous and discontinuous vorticity field [Eq. (A8)] and for a vortex ring bubble [Eq. (A20)].

1. General equations

The equations of the asymmetric components u^a, v^a, u^a at first order are[1]

$$\frac{1}{\overline{r}}[v_{\varphi}^{a(1)}+(\overline{r}u^{a(1)})_{\overline{r}}]=-v^{(0)}K^{(0)}\sin\varphi, \tag{A1}$$

$$\zeta^{(0)}u^{a(1)}+\frac{v^{(0)}}{\overline{r}}v_{\varphi}^{a(1)}+\frac{1}{\overline{r}}p_{\varphi}^{a(1)}=w^{(0)^2}K^{(0)}\sin\varphi, \tag{A2}$$

$$w_{\varphi}^{a(1)}=\mathcal{S}_w^{a(1)}, \tag{A3}$$

$$v^{(0)}u_{\varphi}^{a(1)}-2v^{(0)}v^{a(1)}+p_{\overline{r}}^{a(1)}=-w^{(0)^2}K^{(0)}\overline{r}\cos\varphi, \tag{A4}$$

where

$$\mathcal{S}_w^{a(1)}=\frac{\overline{r}}{v^{(0)}}[-u^{a(1)}w_{\overline{r}}^{(0)}-w^{(0)}v^{(0)}K^{(0)}\sin\varphi]. \tag{A5}$$

As s and τ are parameters in these equations (there are no s-derivative nor τ-derivative), their solution is the same as in the one-time analysis. Following this analysis[1] I define a stream function $\psi^{a(1)}$ by

$$u^{a(1)}=\frac{1}{\overline{r}}\psi_{\varphi}^{a(1)},\quad\text{and}\quad v^{a(1)}=-\psi_{\overline{r}}^{a(1)}+\overline{r}v^{(0)}K^{(0)}\cos\varphi, \tag{A6}$$

and expand it in a Fourier series

$$\psi^{a(1)}=\sum_{n=1}^{\infty}\psi_{n1}^{(1)}\cos n\varphi+\psi_{n2}^{(1)}\sin n\varphi. \tag{A7}$$

The only non-zero Fourier component is $\psi_{11}^{(1)}$ and is given by

$$\frac{\psi_{11}^{(1)}(\overline{r},s,\tau,t)}{K^{(0)}(s,t)v^{(0)}(\overline{r},s,\tau,t)}=\int_0^{\overline{r}}\frac{\int_0^z x\mathcal{D}(x,s,\tau,t)dx}{z[v^{(0)}(z,s,\tau,t)]^2}dz+\int_0^{\overline{r}}z\frac{w^{(0)^2}}{v^{(0)^2}}dz+\frac{\overline{r}^2}{2}, \tag{A8}$$

where

$$\mathcal{D}(\overline{r},s,\tau,t)=v^{(0)^2}-2w^{(0)^2}. \tag{A9}$$

Here, I used the boundary conditions $\psi_{11}^{(1)}(\overline{r}=0)=0$ and $[\psi_{11}^{(1)}]_{\overline{r}}(\overline{r}=0)$. This expression (A8) has been written in this form, without derivatives of the velocity field, so that it is easy to find $\psi_{11}^{(1)}$ if the velocity field is discontinuous. Any expression with derivatives of the velocity field would be more difficult to use because of the contributions of the delta-functions that would be in the integrals. Equation (A8) extends the Callegari and Ting[1] equation of the stream function $\psi_{11}^{(1)}$ to axial core-variations of the leading-order velocity field and to discontinuous vorticity field.

The asymmetric velocity field $(u^{a(1)},v^{a(1)},w^{a(1)})$ and the asymmetric pressure $p^{a(1)}$ depend only on the leading-order velocity field. From Eqs. (A6), (A3), and (A2), it comes

$$u^{a(1)}=u_{11}^{(1)}\sin\varphi, \tag{A10}$$

$$v^{a(1)}=v_{11}^{(1)}\cos\varphi, \tag{A11}$$

$$w^{a(1)}=w_{11}^{(1)}\cos\varphi, \tag{A12}$$

$$p^{a(1)}=p_{11}^{(1)}\cos\varphi, \tag{A13}$$

with

$$u_{11}^{(1)}=-\psi_{11}^{(1)}/\overline{r},$$

$$v_{11}^{(1)}=-(\psi_{11}^{(1)})_{\overline{r}}+K^{(0)}\overline{r}v^{(0)},$$

$$w_{11}^{(1)}=-\psi_{11}^{(1)}w_{\overline{r}}^{(0)}/v^{(0)},$$

$$p_{11}^{(1)}=-\psi_{11}^{(1)}\zeta^{(0)}-\overline{r}K^{(0)}w^{(0)^2}-v^{(0)}v_{11}^{(1)}.$$

2. Discontinuous vortex field

For a vortex with the vorticity inside a vortex tube, the first-order asymmetric part of the condition of continuity of the pressure (4) yields

$$[[p^{a(1)}]] = -\frac{\overline{\delta}^{ta(1)}}{\overline{\delta}^{t(0)}}[[v^{(0)^2}]],$$

where $\overline{\delta}^{ta(1)}$ is the asymmetric part of the first-order thickness of the tube. Here, I used Eq. (13) to have $[[p_{\overline{r}}^{(0)}]] = [[v^{(0)^2}]]/\overline{\delta}^{t(0)}$. The same jump can be found from Eqs. (A13) and (A8). The first-order asymmetric part of the condition of continuity of the normal velocity (7) gives

$$[[u^{a(1)}]] = \frac{\overline{\delta}_{\varphi}^{ta(1)}}{\overline{\delta}^{t(0)}}[[v^{(0)}]],$$

and so

$$[[\psi_{11}^{(1)}]] = \overline{\delta}_{11}^{t(1)}[[v^{(0)}]],$$

where $\overline{\delta}^{ta(1)} = \overline{\delta}_{11}^{t(1)}\cos\varphi$. The same jump can be found from Eq. (A8). It means that Eq. (A8) of $\psi_{11}^{(1)}$ is correct even through the interface. The asymmetric part of the kinematic boundary condition at first order gives

$$\overline{\delta}_{\varphi}^{ta(1)} = \frac{\overline{\delta}^{t(0)}}{v^{(0)}}u^{a(1)}. \tag{A14}$$

Here, all velocity components are taken on the interface $\overline{r} = \overline{\delta}^{t(0)\pm}$. This equation yields

$$\overline{\delta}_{11}^{t(1)} = \psi_{11}^{(1)}/v^{(0)}. \tag{A15}$$

3. Vortex ring bubble

For a vortex bubble, the first order for the asymmetric part of the continuity of pressure (9) is

$$p_{11}^{(1)} + \frac{\overline{\delta}_{11}^{b(1)}}{\overline{\delta}^{b(0)}}v^{(0)^2} = \overline{\Upsilon}K^{(0)}, \tag{A16}$$

where $\overline{\delta}_{11}^{b(1)}$ is the asymmetric part of the first order free-boundary of the bubble. Here, all fields are taken on the free-boundary $\overline{\delta}^{b(0)+}$. This equation is the equation of the thickness $\overline{\delta}_{11}^{b(1)}$ of the bubble. The first order of the asymmetric part of the dynamical equation of the free-boundary (10) is

$$\overline{\delta}_{\varphi}^{ba(1)} = \frac{\overline{\delta}^{b(0)}}{v^{(0)}}u^{a(1)}. \tag{A17}$$

Here, all velocity components are taken on the free-boundary $\overline{r} = \overline{\delta}^{b(0)+}$. This equation yields

$$\overline{\delta}_{11}^{b(1)} = \psi_{11}^{(1)}/v^{(0)}. \tag{A18}$$

This last equation combined with Eqs. (A16) and (A13) gives the following boundary conditions for the equation of $\psi_{11}^{(1)}$

$$\overline{\delta}^{b(0)}(v^{(0)^2} + w^{(0)^2}) + \overline{\Upsilon} = \frac{(\psi_{11}^{(1)})_{\overline{r}}}{K^{(0)}}v^{(0)}, \tag{A19}$$

where I used the second boundary condition $\psi_{11}^{(1)}(\overline{\delta}^{b(0)}) = 0$. The only solution of the equation of $\psi_{11}^{(1)}$ with these two boundary conditions is

$$\frac{\psi_{11}^{(1)}(\overline{r},s,\tau,t)}{K^{(0)}(s,t)v^{(0)}(\overline{r},s,\tau,t)} = \int_{\overline{\delta}^{b(0)}}^{\overline{r}} \frac{\int_{\overline{\delta}^{b(0)}}^{z} x\mathcal{D}(x,s,\tau,t)dx}{z[v^{(0)}(z,s,\tau,t)]^2}dz + \int_{\overline{\delta}^{b(0)}}^{\overline{r}} z\frac{w^{(0)^2}}{v^{(0)^2}}dz + \frac{\overline{r}^2 - (\overline{\delta}^{b(0)})^2}{2} + \overline{\delta}^{b(0)}\overline{\Upsilon}\int_{\overline{\delta}^{b(0)}}^{\overline{r}}\frac{1}{z[v^{(0)}(z,s,\tau,t)]^2}dz. \tag{A20}$$

The behavior of $\int_{\overline{\delta}^{b(0)}}^{\overline{r}} 1/(z[v^{(0)}(z,s,\tau,t)]^2)dz$ at infinity is $\pi\overline{r}/\Gamma$, where I used Hôpital's rule and the behavior of $v^{(0)}$ at infinity.

APPENDIX B: THE FIRST-ORDER SHORT-TIME AXISYMMETRIC DYNAMICS IN THE FILAMENT

In this appendix I give the first-order equations of the short-time axisymmetric dynamics for axial variations in the filament. These equations come from the axisymmetric part at second order. In the two-time scale framework the time-average of these equations gives (Sec. III C) the leading-order normal-time equations in the filament. In the one-time (normal-time) framework the compatibility conditions to these equations also give (Secs. IV and VI) the leading-order one-time equations in the filament. The one-time solution to these equations and its uniqueness is studied in Appendix D.

1. Continuous vorticity field

The first-order compatibility equations of the one-time analysis become the dynamical equations of the axisymmetric part of the first-order relative velocity field. At first order

$$p^{c(1)} = -\int_{\overline{r}}^{\infty}\frac{2v^{(0)}v^{c(1)}}{\overline{r}}d\overline{r}, \tag{B1}$$

and at second order

$$(\overline{r}u^{c(2)})_{\overline{r}} + \overline{r}w_z^{c(1)} = \mathcal{S}^{c(1)}, \tag{B2}$$

$$\frac{\partial v^{c(1)}}{\partial\tau} + \zeta^{c(1)}u^{c(1)} + \zeta^{(0)}u^{c(2)} + w^{c(1)}v_z^{(0)} + w^{(0)}v_z^{c(1)} = \mathcal{S}_v^{c(1)}, \tag{B3}$$

$$\frac{\partial w^{c(1)}}{\partial\tau} + w_{\overline{r}}^{(0)}u^{c(2)} + w_{\overline{r}}^{c(1)}u^{c(1)} + p_z^{c(1)} + w^{c(1)}w_z^{(0)} + w^{(0)}w_z^{c(1)} = \mathcal{S}_w^{c(1)}, \tag{B4}$$

where $\zeta^{c(1)} = (\overline{r}v^{c(1)})_{\overline{r}}/\overline{r}$ is the axisymmetric part of the first-order axial vorticity and

$$\mathcal{S}^{c(1)} = -\frac{\dot{\sigma}^{(0)}}{\sigma^{(0)}}\overline{r} - \frac{\sigma^{(1)}}{\sigma^{(0)}}(\overline{r}u_{\overline{r}}^{c(1)}),$$

$$\mathcal{S}_v^{c(1)} = w^{(0)}v_z^{(0)}\frac{\sigma^{(1)}}{\sigma^{(0)}} - \frac{\partial v^{(0)}}{\partial t} + \overline{\nu}\left(\frac{(\overline{r}v_{\overline{r}}^{(0)})_{\overline{r}}}{\overline{r}} - \frac{v^{(0)}}{\overline{r}^2}\right),$$

$$\mathcal{S}_w^{c(1)} = w^{(0)} w_z^{(0)} \frac{\sigma^{(1)}}{\sigma^{(0)}} + p_z^{(0)} \frac{\sigma^{(1)}}{\sigma^{(0)}} - \frac{\dot{\sigma}^{(0)}}{\sigma^{(0)}} w^{(0)} - \frac{\partial w^{(0)}}{\partial t} + \overline{\nu} \frac{(\overline{r} w_{\overline{r}}^{(0)})_{\overline{r}}}{\overline{r}}.$$

Here, I used the first-order asymmetric field to state that the φ-average of $\zeta^{(1)} u^{(1)}$ is $\zeta^{c(1)} u^{c(1)}$, and that the one of $w_{\overline{r}}^{(1)} u^{(1)}$ is $w_{\overline{r}}^{c(1)} u^{c(1)}$. This system, of unknown $p^{c(1)}$, $v^{c(1)}$, $w^{c(1)}$, and $u^{c(2)}$, is closed.

2. Discontinuous vorticity field

For a vortex with the vorticity inside a vortex tube the interface dynamics has to be found and the previous equations have to be completed. The second-order axisymmetric part of the condition of continuity of the pressure (4) yields

$$[[p^{c(1)}]] = -\frac{\overline{\delta}^{tc(1)}}{\overline{\delta}^{t(0)}} [[v^{(0)^2}]],$$

where $\overline{\delta}^{tc(1)}$ is the axisymmetric part of the first-order interface thickness. Here, I used Eq. (13) to have $[[p_{\overline{r}}^{(0)}]] = [[v^{(0)^2}]]/\overline{\delta}^{t(0)}$. This jump condition means that the expression (B1) of $p^{c(1)}$ is not correct through the interface and has to be replaced by

$$p^{c(1)} = \begin{cases} -\int_{\overline{r}}^{\infty} \frac{2v^{(0)} v^{c(1)}}{\overline{r}} d\overline{r} + \frac{\overline{\delta}^{tc(1)}}{\overline{\delta}^{t(0)}} [[v^{(0)^2}]] & \text{if } \overline{r} < \overline{\delta}^{t(0)}, \\ -\int_{\overline{r}}^{\infty} \frac{2v^{(0)} v^{c(1)}}{\overline{r}} d\overline{r} & \text{if } \overline{r} > \overline{\delta}^{t(0)}. \end{cases}$$

The second-order axisymmetric part of the continuity of the normal velocity (7) yields

$$[[u^{c(2)}]] = \left(\overline{\delta}_z^{t(1)} - \frac{\sigma^{(1)}}{\sigma^{(0)}} \overline{\delta}_z^{t(0)} \right) [[w^{(0)}]] + \overline{\delta}_z^{t(0)} [[w^{c(1)}]] + \overline{\delta}_z^{t(0)} \overline{\delta}^{tc(1)} [[w_{\overline{r}}^{(0)}]]. \quad \text{(B5)}$$

Here, I used the first-order asymmetric field to state that the φ-averages of $v^{(1)} \overline{\delta}_\varphi^{t(1)} / \overline{\delta}^{t(0)}$ and of $u_{\overline{r}}^{(1)} \overline{\delta}^{t(1)}$ are zero.

The axisymmetric part of the kinematic boundary condition at second order gives

$$\frac{\partial \overline{\delta}^{tc(1)}}{\partial \tau} - u^{c(2)} + w^{(0)} \overline{\delta}_z^{tc(1)} + \overline{\delta}_z^{t(0)} w^{c(1)} + w_{\overline{r}}^{(0)} \overline{\delta}_z^{t(0)} \overline{\delta}^{tc(1)} = \mathcal{S}_b^{c(1)}, \quad \text{(B6)}$$

where

$$\mathcal{S}_b^{c(1)} = -\frac{\partial \overline{\delta}^{t(0)}}{\partial t} + w^{(0)} \frac{\sigma^{(1)}}{\sigma^{(0)}} \overline{\delta}_z^{t(0)}.$$

In this equation all velocity components are taken on the interface $\overline{r} = \overline{\delta}^{t(0)\pm}$.

This system, of unknown $p^{c(1)}$, $v^{c(1)}$, $w^{c(1)}$, $u^{c(2)}$, and $\overline{\delta}^{tc(1)}$, is closed.

3. Vortex ring bubble

For a vortex bubble the interface dynamics has to be found and the previous equations have to be completed. The first-order of the axisymmetric part of the continuity of pressure (9) is

$$p^{c(1)} + \overline{\delta}^{bc(1)} p_{\overline{r}}^{(0)} = \overline{P}_{g0} \left(\frac{\mathcal{V}_0^{b(0)}}{\mathcal{V}^{b(0)}} \right)^k \left[\frac{\mathcal{V}_0^{b(1)}}{\mathcal{V}_0^{b(0)}} - \frac{\mathcal{V}^{b(1)}}{\mathcal{V}^{b(0)}} \right]^k + \overline{\Upsilon} \frac{\overline{\delta}^{bc(1)}}{(\overline{\delta}^{b(0)})^2}, \quad \text{(B7)}$$

where all fields are taken on the free-boundary $\overline{\delta}^{b(0)+}$. This equation gives the axisymmetric part of the thickness $\overline{\delta}^{bc(1)}$ of the bubble.

The first order for the axisymmetric part of the dynamical equation (10) of the free-boundary is

$$\frac{\partial \overline{\delta}^{bc(1)}}{\partial \tau} - u^{c(2)} + w^{(0)} \overline{\delta}_z^{bc(1)} + \overline{\delta}_z^{b(0)} w^{c(1)} + w_{\overline{r}}^{(0)} \overline{\delta}_z^{b(0)} \overline{\delta}^{bc(1)} = \mathcal{S}_b^{bc(1)}, \quad \text{(B8)}$$

where

$$\mathcal{S}_b^{bc(1)} = -\frac{\partial \overline{\delta}^{b(0)}}{\partial t} + w^{(0)} \frac{\sigma^{(1)}}{\sigma^{(0)}} \overline{\delta}_z^{b(0)}.$$

In this equation, all velocity components are taken on the free-boundary $\overline{r} = \overline{\delta}^{b(0)+}$. The bubble allows to have a solution of the equation of continuity (B2) in the form

$$u^{c(2)} = \frac{\mathcal{D}^{c(2)}(s,\tau,t)}{\overline{r}} + u^{\omega c(2)},$$

where $u^{\omega c(2)}$ is regular at $\overline{r} = 0$. As the thickness $\overline{\delta}^{bc(1)}$ is given by Eq. (B7), Eq. (B8) is indeed the equation for $\mathcal{D}^{c(2)}$.

This system, of unknown $p^{c(1)}$, $v^{c(1)}$, $w^{c(1)}$, $u^{\omega c(2)}$, $\mathcal{D}^{c(2)}$, and $\overline{\delta}^{bc(1)}$, is closed.

APPENDIX C: UNIQUENESS PROBLEM AT LEADING ORDER

In this appendix I consider the uniqueness problem of the solutions to the leading-order one-time compatibility conditions. These equations are obtained if the short-time-scale derivative is removed from the leading-order short-time axisymmetric equations (13)–(16) for axial variations. Leading-order velocity fields without axial variations and $u^{c(1)} = 0$ are solutions of these leading-order compatibility conditions. As we will see, the study of small perturbations around the solutions without axial variations seems to indicate that they are the unique solutions to these compatibility conditions.

I introduce a small stationary perturbation $(\tilde{u}^{c(1)}, \tilde{v}^{(0)}, \tilde{w}^{(0)}, \tilde{p}^{(0)}, \tilde{\psi}^{c(1)}, \tilde{\mathcal{K}}^{(0)})$ of a flow without axial variations, denoted by $(\underline{u}^{c(1)}=0,\underline{v}^{(0)},\underline{w}^{(0)},\underline{p}^{(0)},\underline{\psi}^{c(1)},\underline{\mathcal{K}}^{(0)})$. At first order in its amplitude, the perturbation satisfies

$$\tilde{p}^{(0)}=-\int_{\bar{r}}^{\infty}\frac{2\underline{v}^{(0)}\tilde{v}^{(0)}}{\bar{r}}\,d\bar{r}, \tag{C1}$$

$$(\bar{r}\tilde{u}^{c(1)})_{\bar{r}}+\bar{r}\tilde{w}_z^{(0)}=0, \tag{C2}$$

$$\underline{\zeta}^{(0)}\tilde{u}^{c(1)}+\underline{w}^{(0)}\tilde{v}_z^{(0)}=0, \tag{C3}$$

$$\underline{w}_{\bar{r}}^{(0)}\tilde{u}^{c(1)}+\tilde{p}_z^{(0)}+\underline{w}^{(0)}\tilde{w}_z^{(0)}=0, \tag{C4}$$

or the equivalent system

$$\underline{\psi}^{c(1)}\tilde{\mathcal{K}}_z^{(0)}=\underline{\mathcal{K}}_y^{(0)}f, \tag{C5}$$

$$4y\,\underline{\psi}_y^{c(1)}f_{yy}+\frac{1}{y}\underline{\mathcal{K}}^{(0)}\tilde{\mathcal{K}}_z^{(0)}-4y\,\underline{\psi}_{yyy}^{c(1)}f=0, \tag{C6}$$

where $f=\tilde{\psi}_z^{c(1)}$. If $\underline{\psi}_y^{c(1)}=0$ then Eq. (C5) yields $\tilde{\psi}_z^{c(1)}\underline{\mathcal{K}}_y^{(0)}=0$ and so $\tilde{\psi}_z^{c(1)}=0$ is the only physical solution. Equation (C6) then yields $\tilde{\mathcal{K}}_z^{(0)}=0$ as $\underline{\mathcal{K}}^{(0)}\neq 0$. So the only possible stationary perturbation is without axial variations. If $\underline{\psi}_y^{c(1)}\neq 0$, Eqs. (C5)–(C6) yield

$$f''+G(y)f=0, \tag{C7}$$

$$\lim_{y\to+\infty} f=0,$$

$$f'(0)=0,$$

$$\tilde{\mathcal{K}}_z^{(0)}=f\underline{\mathcal{K}}_y^{(0)}/\underline{\psi}_y^{c(1)},$$

with

$$G=\left[\frac{\underline{\mathcal{K}}^{(0)}\underline{\mathcal{K}}_y^{(0)}}{4y^2}-\underline{\psi}_y^{c(1)}\underline{\psi}_{yyy}^{c(1)}\right]\bigg/\ \underline{\psi}_y^{c(1)^2}.$$

For most flows $(\underline{\psi}^{c(1)},\mathcal{K}^{(0)})$, $f=0$ seems to be the unique solution of the linear equation (C7) and so the only possible stationary perturbation seems to be without axial variations. The velocity fields without axial variations seems to be isolated solutions of the leading-order compatibility conditions. Souza[12] used a standard comparison principle for quasi-linear elliptic operators and proved that there are no other stationary solutions of the Bragg–Hawthorne equation (22) than the solutions without axial variations. Klein and Ting[34] assumed that these compatibility conditions have stationary solutions with axial variations and derived the equations of evolution of these fields in a one-time analysis on the normal-time scale. Unfortunately, as was pointed out by Souza (private communication), no field with axial variations and without axial velocity at infinity is solution of the leading-order compatibility conditions.

For a vortex with the vorticity inside a vortex tube, it also comes $\overline{\delta^{t(0)}}(s,t)=\overline{\delta^{t(0)}}(t)$, and for a vortex bubble it comes $\overline{\delta^{b(0)}}(s,t)=\overline{\delta^{b(0)}}(t)$.

APPENDIX D: UNIQUENESS PROBLEM AT FIRST ORDER

In this appendix I consider the uniqueness problem of the solutions to the one-time compatibility conditions at next order. These equations are obtained if the short-time scale derivative is removed from the first-order short-time axisymmetric equations (B1)–(B4) for the axial variations. Assuming that the compatibility conditions at first order have the only solution without axial variations (as suggested in Appendix C) it is found in this appendix that these compatibility conditions at second order also have a unique solution. Here, this solution is given and proves to be the one introduced by Margerit[17] to generalize the Callegari and Ting theory[1] at next order.

I define

$$\mathsf{s}=\frac{\dot{\sigma}^{(0)}}{\sigma^{(0)}}-\frac{\dot{S}^{(0)}}{S^{(0)}},$$

$$\tilde{z}=\int_0^s\sigma\,\mathsf{s}\,ds',$$

$$\chi^{(2)}=\frac{1}{\mathsf{s}}\left(u^{c(2)}+\frac{1}{2}\bar{r}\frac{\dot{S}^{(0)}}{S^{(0)}}\right),$$

$$\beta^{(1)}=w^{c(1)}+\tilde{z}.$$

The subtraction of Eqs. (34)–(35) from Eqs. (B1)–(B4) yields

$$p^{c(1)}=-\int_{\bar{r}}^{\infty}\frac{2v^{(0)}v^{c(1)}}{\bar{r}}\,d\bar{r}, \tag{D1}$$

$$(\bar{r}\chi^{(2)})_{\bar{r}}+\bar{r}\beta_{\tilde{z}}^{(1)}=0, \tag{D2}$$

$$\zeta^{(0)}\chi^{(2)}+w^{(0)}v_{\tilde{z}}^{c(1)}=0, \tag{D3}$$

$$w_{\bar{r}}^{(0)}\chi^{(2)}+p_{\tilde{z}}^{c(1)}+w^{(0)}\beta_{\tilde{z}}^{(1)}=0, \tag{D4}$$

or the equivalent system

$$\psi^{c(1)}\mathcal{K}_{\tilde{z}}^{c(1)}=\mathcal{K}_y^{(0)}f, \tag{D5}$$

$$4y\,\psi_y^{c(1)}f_{yy}+\frac{1}{y}\mathcal{K}^{(0)}\mathcal{K}_{\tilde{z}}^{c(1)}-4y\,\psi_{yyy}^{c(1)}f=0, \tag{D6}$$

where

$$f=\psi_{\tilde{z}}^{c(2)},$$

$$\chi^{(2)}=-\frac{1}{\bar{r}}\psi_{\tilde{z}}^{c(2)},$$

$$\beta^{c(1)}=\frac{1}{\bar{r}}\psi_{\bar{r}}^{c(2)},$$

$$\mathcal{K}^{c(1)}=\bar{r}v^{c(1)}.$$

As the linear operator of the systems (D1)–(D4) and (D5)–(D6) is the same as the one of Eqs. (C1)–(C4) and (C5)–(C6), the unique solution of these systems is $f=0$ and $\mathcal{K}_{\tilde{z}}^{c(1)}=0$. The unique stationary solutions of the compatibility conditions for the first-order axisymmetric field are

$$v^{c(1)}(\overline{r},s,t)=v^{u(1)}(\overline{r},t),$$

$$w^{c(1)}(\overline{r},s,t)=w^{u(1)}(\overline{r},t)+\int_0^s\left(-\dot{\sigma}^{(0)}+\sigma^{(0)}\frac{\dot{S}^{(0)}}{S^{(0)}}\right)ds',$$

$$u^{c(2)}(\overline{r},s,t)=-\frac{1}{2}\overline{r}\frac{\dot{S}^{(0)}}{S^{(0)}},$$

where the evolution of $(v^{u(1)},w^{u(1)})$ in the normal-time scale can be found from the axisymmetric equations at third-order.[17]

For a vortex with the vorticity inside a vortex tube, the subtraction of Eq. (36) from Eq. (B6) written without the short-time scale derivative yields $\overline{\delta}_z^{tc(1)}=0$, i.e., $\overline{\delta}^{tc(1)}(s,t)=\overline{\delta}^{tc(1)}(t)$.

[1] A. Callegari and L. Ting, ''Motion of a curved vortex filament with decaying vortical core and axial velocity,'' SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **35**, 148 (1978).
[2] R. Klein and O. Knio, ''Asymptotic vorticity structure and numerical simulation of slender vortex filaments,'' J. Fluid Mech. **284**, 257 (1995).
[3] J. Marshall, ''A general theory of curved vortices with circular cross-section and variable core area,'' J. Fluid Mech. **229**, 311 (1991).
[4] J. Marshall, ''The effect of axial pressure gradient on axisymmetrical and helical vortex waves,'' Phys. Fluids A **5**, 588 (1993); erratum: Phys. Fluids A **5**, 1857 (1993).
[5] A. Leonard, ''Nonlocal theory of area-varying waves on axisymmetric vortex tubes,'' Phys. Fluids **6**, 765 (1994).
[6] T. Lundgren and W. Ashurts, ''Area-varying waves on curved vortex tubes with application to vortex breakdown,'' J. Fluid Mech. **200**, 283 (1989).
[7] M. Melander and F. Hussain, ''Core dynamics on a vortex column,'' Fluid Dyn. Res. **13**, 1 (1994).
[8] S. Arendt, D. C. Fritts, and Ø. Andreassen, ''The initial value problem for Kelvin vortex waves,'' J. Fluid Mech. **344**, 181 (1997).
[9] V. F. Kopiev and S. Chernyshev, ''Vortex ring eigen-oscillations as a source of sound,'' J. Fluid Mech. **341**, 19 (1997).
[10] S. Widnall and J. Sullivan, ''On the stability of vortex rings,'' Proc. R. Soc. London, Ser. A **332**, 335 (1973).
[11] D. Margerit and J.-P. Brancher, ''Motion and oscillations of a circular perturbed vortex ring,'' C. R. Acad. Sci., Ser. IIb: Mec., Phys., Chim., Astron. **328**, 1 (2000).
[12] M. Souza, ''Instabilities of rotating and unsteady flows,'' Ph.D. thesis, Darwin College, University of Cambridge, 1998.
[13] D. Margerit and J.-P. Brancher, ''Asymptotic expansions of the Biot–Savart law for a slender vortex with core variation,'' J. Eng. Math. **40**, 297 (2001).
[14] P. Genoux, ''Étude asymptotique du mouvement et des oscillations d'un tore de vapeur. Modélisation d'un jet cavitant oscillant,'' Ph.D. thesis, Université de Paris 6, 1988.
[15] P. Genoux and G. Chahine, ''Équilibre statique et dynamique d'un tore de vapeur tourbillonnaire,'' J. Mec. Theor. Appl. **2**, 829 (1983).
[16] G. Chahine and P. Genoux, ''Collapse of a caviting vortex ring,'' J. Fluids Eng. **105**, 400 (1983).
[17] D. Margerit, in *The Complete First Order Expansion of a Slender Vortex Ring. IUTAM Symposium on Dynamics of Slender Vortices*, edited by E. Krause and K. Gersten (Kluwer Academic, Aachen, 1997), pp. 45–54.
[18] J.-Z. Wu, ''A theory of three-dimensional interfacial dynamics,'' Phys. Fluids **7**, 2375 (1995).
[19] S. Widnall and C. Tsai, ''Motion and decay of a vortex in a nonuniform stream,'' Philos. Trans. R. Soc. London, Ser. A **287**, 273 (1977).
[20] S. Leibovich, ''Weakly nonlinear waves in rotating fluids,'' J. Fluid Mech. **42**, 803 (1970).
[21] L. Ting and C. Tung, ''Motion and decay of a vortex in a nonuniform stream,'' Phys. Fluids **8**, 1039 (1965).
[22] M. Gunzburger, ''Long time behavior of a decaying vortex,'' Z. Angew. Math. Mech. **53**, 751 (1973).
[23] L. Ting and R. Klein, *Viscous Vortical Flows (Monograph)*, Lecture Notes in Physics (Springer, Berlin, 1991), pp. 181–185.
[24] R. Klein and L. Ting, ''Theoretical and experimental studies of slender vortex filaments,'' Appl. Math. Lett. **8**, 45 (1995).
[25] R. Klein, O. Knio, and L. Ting, ''Representation of core dynamics in slender vortex filament simulations,'' Phys. Fluids **8**, 2415 (1996).
[26] V. F. Kopiev and S. Chernyshev, ''Small disturbances of steady vortices,'' Fluid Dyn. **26**, 719 (1991).
[27] V. F. Kopiev and S. Chernyshev, ''Long-wave instability of a vortex ring,'' Fluid Dyn. **30**, 864 (1995).
[28] Lord Kelvin, ''Vibration of a columnar vortex,'' Philos. Mag. **10**, 152 (1880).
[29] M. Lessen, P. Singh, and F. Paillet, ''The stability of a trailing line vortex. Part I. Inviscid theory,'' J. Fluid Mech. **63**, 753 (1974).
[30] T. Loiseleux, J. Chomaz, and P. Huerre, ''The effect of swirl on jets and wakes: Linear instability of the Rankine vortex with axial flow,'' Phys. Fluids **10**, 1120 (1998).
[31] C. Olendraru, A. Sellier, M. Rossi, and P. Huerre, ''Inviscid instability of the Batchelor vortex: Absolute-convective transition and spatial branches,'' Phys. Fluids **11**, 1805 (1999).
[32] T. Lundgren, ''Strained spiral vortex model for turbulent fine structure,'' Phys. Fluids **25**, 2193 (1982).
[33] O. Knio and R. Klein, ''Improved thin-tube models for slender vortex simulations,'' J. Comput. Phys. **163**, 68 (2000).
[34] R. Klein and L. Ting, ''Vortex filaments with axial core structure variation,'' Appl. Math. Lett. **5**, 99 (1992).