THE COMPLETE FIRST ORDER EXPANSION OF A SLENDER VORTEX RING

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Abstract. Equations for the axisymmetric part of the velocity field and for the equation of motion of a *non circular* slender vortex ring are given at first order. This is the correction to the known leading order given by Callegari and Ting [2].

I. DEFINITIONS AND NOTATIONS

The length scales of the vortex ring that are different from its thickness δ, for example : the radius of curvature, the ring length, are of the same order L with $\delta/L = O(\epsilon) << 1$. The central curve is described parametricaly with the use of a function $\mathbf{X} = \mathbf{X}(s,t)$. A local curvilinear co-ordinate system (r, φ, s) , with a frame $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{t})$, is introduced near this central curve [2]. There is an *outer problem* defined by the *outer limit* : $\epsilon \to 0$ with r fixed, which describes the situation far from the central line and an *inner problem* defined by the *inner limit* : $\epsilon \to 0$ with $\overline{r} = r/\epsilon$ fixed, which describes the situation near the central line.

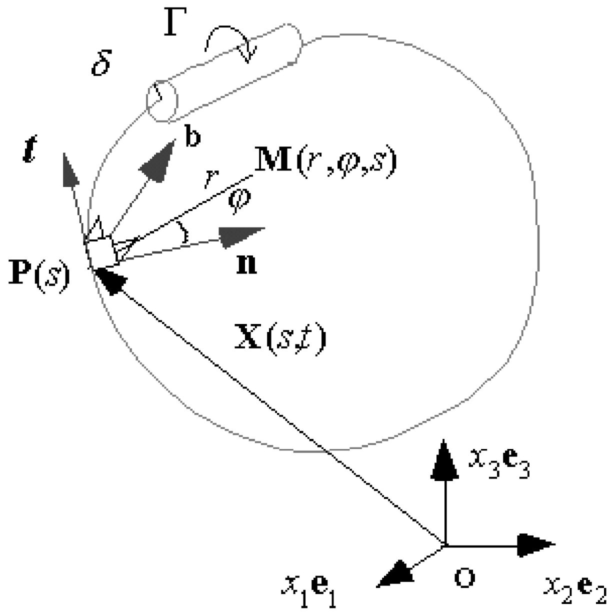

FIG. 1. The central curve and the local co-ordinates of the vortex ring.

The change between Cartesian co-ordinates $\mathbf{M}(x_1, x_2, x_3)$ and local co-ordinates $\mathbf{M}(r, \varphi, s)$ satisfies :

$$\mathbf{x} = \mathbf{OM} = \mathbf{X}(s,t) + r\mathbf{e}_r(\varphi, s, t)$$

The variable

$$\sigma(s,t) = |\mathbf{X}_s| \tag{1}$$

is introduced, where $|\ |$ is the usual norm of $\mathbf{R^3}$. The Frenet formulas are

$$\begin{array}{ll} \mathbf{X}_s = \sigma \mathbf{t} & \mathbf{t}_s = \sigma K \mathbf{n} \\ \mathbf{n}_s = \sigma(T\mathbf{b} - K\mathbf{t}) & \mathbf{b}_s = -\sigma T \mathbf{n} \end{array} \tag{2}$$

where T the local torsion of $\mathcal{C}$ and K the local curvature of $\mathcal{C}$. Notice that here and throughout this paper, the differentiation $\partial f / \partial x$ of a function f with respect to its variable x is denoted f_x ; $\times$ is the cross-product and $\cdot$ is the dot-product.The polar vectors $(\mathbf{e}_r, \mathbf{e}_\theta)$ are

$$\mathbf{e}_r(\varphi, s) = +\mathbf{n}(s)\cos(\varphi) + \mathbf{b}(s)\sin(\varphi) \tag{3}$$

$$\mathbf{e}_\theta(\varphi, s) = -\mathbf{n}(s)\sin(\varphi) + \mathbf{b}(s)\cos(\varphi) \tag{4}$$

The small parameter ϵ is defined by $\epsilon = \delta_0/L = \delta(t=0)/L$.

Dimensionless variables :

$$\begin{array}{ll} \mathbf{X}^* = \mathbf{X}/L & S^* = S/L \\ K^* = LK & T^* = LT \\ \delta^* = \delta/L & t^* = t/(L^2/\Gamma) \\ \mathbf{v}^* = \mathbf{v}/(\Gamma/L) & \boldsymbol{\omega}^* = \boldsymbol{\omega}/(\Gamma/L^2) \\ r^* = r/L & \sigma^* = \sigma/L \end{array}$$

are introduced, where S is the length of the ring and Γ is its circulation. Here, $\mathbf{v}$ and $\boldsymbol{\omega}$ are respectively the velocity and the vorticity fields. From here on, all quantities are dimensionless and the asterisks are omitted. The Reynolds number R_e is defined by $R_e = \Gamma/\nu$ where ν is the kinematic viscosity of the fluid. Let us define the number α such that $R_e^{-1/2} = \alpha\epsilon$. Both inviscid : $\alpha = 0$ and viscous : $\alpha = O(1)$ vortex rings are studied. The velocity is decomposed as follows :

$$\mathbf{v}(r, \varphi, s, t, \epsilon) = \dot{\mathbf{X}}(s, t, \epsilon) + \mathbf{V}(r, \varphi, s, t, \epsilon) \tag{5}$$

where

$$\mathbf{V} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{t} \tag{6}$$

and

$$\dot{\mathbf{X}} = \frac{\partial \mathbf{X}}{\partial t} \tag{7}$$

The following forms are chosen for the inner expansions of the velocity field :

$$\begin{array}{lllll} u^{inn} & = & & u^{(1)}(\overline{r}, \varphi, s, t) & +... \\ v^{inn} & = & \epsilon^{-1}v^{(0)}(\overline{r}, s, t) & + \ v^{(1)}(\overline{r}, \varphi, s, t) & +... \\ w^{inn} & = & \epsilon^{-1}w^{(0)}(\overline{r}, s, t) & + \ w^{(1)}(\overline{r}, \varphi, s, t) & +... \end{array} \tag{8}$$

with an expression of the central curve of the form :

$$\mathbf{X} = \mathbf{X}^{(0)}(s,t) + \epsilon \mathbf{X}^{(1)}(s,t) + ... \tag{9}$$

II. LIMIT OF $\mathbf{v}^{inn}$ AT $\overline{r} \to \infty$ UP TO ORDER ϵ THROUGH BIOT AND SAVART LAW

Let us have a vorticity field of the form :

$$\boldsymbol{\omega} = \frac{1}{\epsilon^2} \boldsymbol{\omega}^{(0)}(\overline{r}, \varphi, s) \tag{10}$$

The Biot and Savart law is given on local co-ordinates by the formula :

$$\mathbf{v}(r, \varphi, s, t, \epsilon) \tag{11}$$

$$= \frac{1}{4\pi} \iiint \frac{\epsilon^2 \boldsymbol{\omega}(\overline{r}', \varphi', s', t, \epsilon) \times [(\mathbf{X}(s,t,\epsilon) + r\mathbf{e}_r(\varphi, s, t, \epsilon)) - (\mathbf{X}(s', t, \epsilon) + \epsilon \overline{r}' \mathbf{e}_r')]}{|(\mathbf{X}(s,t,\epsilon) + r\mathbf{e}_r(\varphi, s, t, \epsilon)) - (\mathbf{X}(s', t, \epsilon) + \epsilon \overline{r}' \mathbf{e}_r')|} h_3' \overline{r}' d\overline{r}' d\varphi' ds' \tag{12}$$

where

$$h_3' = \sigma(s', t)(1 - K(s', t)\epsilon \overline{r}' \cos(\varphi')). \tag{13}$$

Next, in this section, s will be an arc length parameter.
The outer expansion of velocity is :

$$\mathbf{v}^{out}(r, \varphi, s, \epsilon) = \mathbf{v}^{out^{(0)}}(r, \varphi, s) + \epsilon \mathbf{v}^{out^{(1)}}(r, \varphi, s) + O(\epsilon^2).$$

If

$$\iint (\boldsymbol{\omega} - [\boldsymbol{\omega} \cdot \mathbf{t}]\, \mathbf{t})\, \overline{r} d\overline{r} d\varphi = 0 \tag{14}$$

one obtains :

$$\mathbf{v}^{out(0)}(r, \varphi, s, \epsilon) = \frac{1}{4\pi} \int_{\mathcal{C}} \frac{\mathbf{t}(s') \times (\mathbf{x} - \mathbf{X}(s'))}{|(\mathbf{x} - \mathbf{X}(s')|^3} ds' \tag{15}$$

$$\begin{aligned}
&\mathbf{v}^{out(1)}(r, \varphi, s) \\
&= \frac{1}{4\pi} \iiint \frac{\boldsymbol{\omega}'^{(0)} \times (\mathbf{x} - \mathbf{X}')}{\left|\mathbf{x} - \mathbf{X}'\right|^3} \overline{r}'^2 K(s') \cos(\varphi') d\overline{r}' d\varphi' ds' \\
&- \frac{1}{4\pi} \iiint 3 \frac{[\boldsymbol{\omega}'^{(0)} \times (\mathbf{x} - \mathbf{X}')][\mathbf{e}_r' \cdot (\mathbf{x} - \mathbf{X}')]}{\left|\mathbf{x} - \mathbf{X}'\right|^5} \overline{r}'^2 d\overline{r}' d\varphi' ds' \\
&- \frac{1}{4\pi} \iiint \frac{\mathbf{e}_r' \times \boldsymbol{\omega}'^{(0)}}{\left|\mathbf{x} - \mathbf{X}'\right|^3} \overline{r}'^2 d\overline{r}' d\varphi' ds'
\end{aligned} \tag{16}$$

with :

$$\mathbf{x} = \mathbf{X}(s, t) + r\mathbf{e}_r(\varphi, s, t) \tag{17}$$

Thus at leading order in outer co-ordinates, the velocity field exactly correspond to the Dirac delta distribution

$\delta_C \mathbf{t}$ on the central line.
In case

$$\boldsymbol{\omega}^{(0)} = \omega_2(\overline{r})\mathbf{e}_\theta + \omega_3(\overline{r})\mathbf{t}, \tag{18}$$

when $r = \epsilon\overline{r}$ is put in $\mathbf{v}^{out}(r \to 0, \varphi, s)$, one obtains :

$$\begin{aligned}\mathbf{v}^{inn}(\overline{r} \to \infty, \varphi, s) = \frac{1}{\epsilon}\mathbf{v}^{inn(0)}(\overline{r} \to \infty, \varphi, s) + \ln\epsilon\mathbf{v}^{inn(01)}(\overline{r} \to \infty, \varphi, s) + \mathbf{v}^{inn(1)}(\overline{r} \to \infty, \varphi, s) \\ +\epsilon\ln\epsilon\mathbf{v}^{inn(12)}(\overline{r} \to \infty, \varphi, s) + \epsilon\mathbf{v}^{inn(2)}(\overline{r} \to \infty, \varphi, s) + O(\epsilon^2\ln\epsilon)\end{aligned} \tag{19}$$

with :

$$\mathbf{v}^{inn(0)}(\overline{r} \to \infty, \varphi, s) = \frac{1}{2\pi}\frac{\mathbf{e}_\theta}{\overline{r}} + \frac{\mathbf{I}_1}{\overline{r}^2} + O(\frac{1}{\overline{r}^3}) \tag{20}$$

$$\mathbf{v}^{inn(01)}(\overline{r} \to \infty, \varphi, s) = -\frac{K}{4\pi}\mathbf{b} \tag{21}$$

$$\mathbf{v}^{inn(1)}(\overline{r} \to \infty, \varphi, s) = \frac{K}{4\pi}\left[\ln\frac{S}{\overline{r}} - 1\right]\mathbf{b} + \frac{K}{4\pi}\cos\varphi\mathbf{e}_\theta + \mathbf{A} + \frac{\mathbf{I}_2}{\overline{r}} + O(\frac{1}{\overline{r}^2}) \tag{22}$$

$$\mathbf{v}^{inn(12)}(\overline{r} \to \infty, \varphi, s) = \mathbf{I}_3 + \mathbf{I}_5\overline{r} \tag{23}$$

$$\mathbf{v}^{inn(2)}(\overline{r} \to \infty, \varphi, s) = (\mathbf{I}_3 + \mathbf{I}_5\overline{r})\ln\overline{r} + (\mathbf{I}_6 + \mathbf{E}_2(\varphi, s))\,\overline{r} + \mathbf{I}_4 + \mathbf{E}_1(s) \tag{24}$$

$$\mathbf{E}_2(\varphi, s) = \frac{1}{4\pi}\left(\mathbf{B}(\varphi, s) - 3\mathbf{C}(\varphi, s)\right) \tag{25}$$

where expressions of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{E}_i$, $\mathbf{I}_i$ (i=1...6) are given in Appendix.
This expression (11) can be compared with that of Fukumoto and Miyazaki [4] (page 373) and Callegari et Ting [2] (page 173). It is the same all but here $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and order ϵ are given. Besides here the derivation was performed in an algorithmic way with formal calculus (Maple) and with the matched asymptotic expansion of singular integral method following François [3] or Bender and Orszag [1].
Let us notice that the same result would have be obtained if $\overline{r} \to \infty$ were put in the inner expansion of Biot and Savart law [6].
This result will be used in the following when the asymptotic matching will be performed.

III. RESULTS AT ORDER 0

Callegari and Ting [2] considered the case where $v^{(0)}, w^{(0)}$ are independent of s so that some compatibility conditions are satisfied. They deduced the following equations for $v^{(0)}, w^{(0)}$ from Navier Stokes second order equations :

$$\overline{r}\frac{\partial v^{(0)}(\overline{r},t)}{\partial t} - \alpha^2\frac{\partial v^{(0)}(\overline{r},t)}{\partial\overline{r}} + \frac{\alpha^2}{\overline{r}}v^{(0)}(\overline{r},t) - \alpha^2\overline{r}\frac{\partial^2 v^{(0)}(\overline{r},t)}{\partial\overline{r}^2} - \frac{1}{2}\overline{r}\frac{\partial\overline{r}v^{(0)}(\overline{r},t)}{\partial\overline{r}}\frac{\dot{S}^{(0)}}{S^{(0)}} = 0 \tag{26}$$

$$\overline{r}\frac{\partial w^{(0)}(\overline{r},t)}{\partial t} - \alpha^2\frac{\partial w^{(0)}(\overline{r},t)}{\partial\overline{r}} - \alpha^2\overline{r}\frac{\partial^2 w^{(0)}(\overline{r},t)}{\partial\overline{r}^2} - \frac{1}{2}\overline{r}^4\left(\frac{w^{(0)}(\overline{r},t)}{\overline{r}^2}\right)_{\overline{r}}\frac{\dot{S}^{(0)}}{S^{(0)}} = 0 \tag{27}$$

where $S^{(0)}$ is the length of the ring.
Through matching, they found the following equation for $\mathbf{X}^{(0)}(s,t)$:

$$\dot{\mathbf{X}}^{(0)} - (\dot{\mathbf{X}}^{(0)} \cdot \mathbf{t})\mathbf{t} = (\frac{K^{(0)}}{4\pi}\left[\ln\frac{S^{(0)}}{\epsilon} - 1\right] + K^{(0)}C^{*})\mathbf{b} + \mathbf{A} - (\mathbf{A} \cdot \mathbf{t})\mathbf{t} \tag{28}$$

where

$$C^{*}(t) = \frac{1}{4\pi}\left\{+\frac{1}{2} + \lim_{\overline{r}\to\infty}\overline{r}\left(4\pi^2\int_0^{\overline{r}}\xi\left(v^{(0)}\right)^2 d\xi - \ln(\overline{r})\right) - 8\pi^2\int_0^{\infty}\xi(w^{(0)})^2 d\xi\right\} \tag{29}$$

$$\lambda(s,\overline{s},t) = \int_s^{s+\overline{s}} \sigma^{(0)}(s^{*},t)ds^{*} \tag{30}$$

$$\mathbf{A}(s,t) \tag{31}$$

$$= \frac{1}{4\pi}\int_{-\pi}^{+\pi}\left[-\sigma^{(0)}(s+\overline{s},t)\frac{\mathbf{t}^{(0)}(s+\overline{s},t)\times(\mathbf{X}^{(0)}(s+\overline{s},t)-\mathbf{X}^{(0)}(s,t))}{\left|\mathbf{X}^{(0)}(s+\overline{s},t)-\mathbf{X}^{(0)}(s,t)\right|^3} - \frac{K^{(0)}(s,t)}{2}\frac{\mathbf{b}^{(0)}(s,t)\sigma^{(0)}(s+\overline{s},t)}{\left|\lambda^{(0)}(s,\overline{s},t)\right|}\right]d\overline{s} \tag{32}$$

IV. RESULTS AT ORDER 1

In the same way that first order Navier Stokes equations give compatibility equations for $v^{(0)}, w^{(0)}$, second order Navier Stokes equations give compatibility equations for the axisymmetric part $v_c^{(1)}, w_c^{(1)}$ of $v^{(1)}, w^{(1)}$. These equations are automatically satisfied if $v_c^{(1)}$ is assumed to be independent of s and if $w_c^{(1)}$ is such that :

$$\frac{\partial w_c^{(1)}(\overline{r},s,t)}{\partial s} = -\dot{\sigma}^{(0)} + a(s,t)\sigma^{(0)} \tag{33}$$

$$a(s,t) = \dot{S}^{(0)}/S^{(0)} \tag{34}$$

We write :

$$w_c^{(1)}(\overline{r},s,t) = w_{cc}^{(1)}(s,t) + w_c^{(1)}(\overline{r},t) \tag{35}$$

Equations for $v_c^{(1)}, w_c^{(1)}$ can be extracted from third order Navier Stokes equations. We did this with the use of symbolic calculus (on maple) in the following way : we obtained third order Navier Stokes equations, then we carried out the φ-average and s-average of the axial and circumferential components of the third order momentum equations using the third order continuity equation to eliminate $u_c^{(2)}$. Lots of terms vanished and we found equations for $v_c^{(1)}, w_c^{(1)}$. Note that Fukumoto and Miyazaki ([4] page 378) postulated $v_c^{(1)} = 0$, kept $w_c^{(1)}$ dependent of s, and did not have equation for $w_c^{(1)}$. The following equations are obtained :

$$S^{(0)}\left(\frac{\partial v_c^{(1)}}{\partial t} - \alpha^2\left[\frac{1}{\overline{r}}(\overline{r}v_c^{(1)})_{\overline{r}}\right]_{\overline{r}}\right) - \frac{1}{2}\dot{S}^{(0)}\left(\overline{r}v_c^{(1)}\right)_{\overline{r}} = \frac{\overline{r}}{2}\left(\dot{S}^{(1)} - S^{(1)}\frac{\dot{S}^{(0)}}{S^{(0)}}\right)\zeta^{(0)} \tag{36}$$

where

$$S^{(1)} = \int_0^{2\pi} \sigma^{(1)} ds. \tag{37}$$

$$S^{(0)} \left(\frac{\partial w_c^{(1)}}{\partial t} - \alpha^2 \left[\frac{1}{\overline{r}} \left(\overline{r} \left(w_c^{(1)} \right)_{\overline{r}} \right)_{\overline{r}} \right]_{\overline{r}} \right) - \frac{1}{2} \dot{S}^{(0)} \overline{r}^3 \left(\frac{w_c^{(1)}}{\overline{r}^2} \right)_{\overline{r}} = \frac{\overline{r}^3}{2} (w^{(0)}/\overline{r}^2)_{\overline{r}} \left(\dot{S}^{(1)} - S^{(1)} \frac{\dot{S}^{(0)}}{S^{(0)}} \right)$$
$$+ \frac{1}{4\pi} \left(\ln(\frac{S}{\epsilon}) + C - 4\pi \overline{r} v^{(0)} \right) \int_0^{2\pi} K^{(0)} \mathbf{A}_s(s,t) \cdot \mathbf{b}^{(0)} ds \tag{38}$$
$$- \int_0^{2\pi} \sigma^{(0)} \mathbf{A}(s,t) \cdot \mathbf{t}^{(0)} ds - \int_0^{2\pi} \sigma^{(0)} \frac{\partial w_{cc}^{(1)}(s,t)}{\partial t} ds - \frac{\dot{S}^{(0)}}{S^{(0)}} \int_0^{2\pi} \sigma^{(0)} w_{cc}^{(1)}(s,t) ds$$

The left hand side of these equations are the same for $v_c^{(1)}(\overline{r},t)$ and $w_c^{(1)}(\overline{r},t)$ than for $v^{(0)}(\overline{r},t)$ and $w^{(0)}(\overline{r},t)$. We may notice that even if initially $w_c^{(1)}(\overline{r},0) = 0$, the right hand side terms will induce $w_c^{(1)}(\overline{r},0) \neq 0$ when $t \neq 0$.
These equations for $v_c^{(1)}, w_c^{(1)}$ are linked to $\mathbf{X}^{(1)}(s,t)$, so an equation for $\mathbf{X}^{(1)}$ is needed to have a closed system of equations for the first order solutions $v_c^{(1)}, w_c^{(1)}$ and $\mathbf{X}^{(1)}$. The best attempt to find this equation is by Fukumoto and Miyazaki ([4] page 382), where contribution from Navier Stokes equations up to order ϵ^1 have been found in order to performed the asymptotic matching. We may note that in their expression the term due to first order curvature $K^{(1)}$ is missing. Moreover, their expression for $\mathbf{X}^{(1)}$ is not complete, as local and non-local (named $\mathbf{Q}$ in [4]) contribution from Biot and Savart integral are given only up to order ϵ^0 in Fukumoto and Miyazaki [4] (page 373) and Callegari and Ting [2] (page 173), while order ϵ^1 is obviously needed to obtain complete and correct equation for $\mathbf{X}^{(1)}$. The complete expression, up to ϵ order, was performed in section 2. The matching is then done and lacking terms in equation of $\mathbf{X}^{(1)}$ are found :

$$\dot{\mathbf{X}}^{(1)} - (\dot{\mathbf{X}}^{(1)} \cdot \mathbf{t})\mathbf{t} = \left\{ C_2^* - \frac{1}{8\pi} K^{(1)}(s,t) + \frac{1}{4\pi} K^{(1)}(s,t) \ln\epsilon - \frac{m}{4\pi} K_s \left[3\ln\epsilon + 3 + \frac{5}{6} - 3\ln S \right] \right\} \mathbf{n}$$
$$+ \left\{ C_1^* + \frac{m}{4\pi} KT \left[\ln\epsilon + \frac{5}{6} - \ln S \right] \right\} \mathbf{b} + \mathbf{E}_1 - (\mathbf{E}_1 \cdot \mathbf{t})\mathbf{t} \tag{39}$$

$$C_1^* = \pi \int_0^{\infty} \xi v^{(0)}(\xi,t) Hs_{12}^{(2)}(\xi,t) d\xi \tag{40}$$

$$C_2^* = \pi \lim_{\overline{r}\to\infty} \left(\int_0^{\overline{r}} \xi v^{(0)}(\xi,t) Hs_{11}^{(2)}(\xi,t) d\xi - \frac{1}{4} \frac{K^{(1)}(s,t)}{\pi^2} \ln\overline{r} \right) \tag{41}$$

$$Hs_{12}^{(2)} = 2 \frac{\xi}{v^{(0)}(\xi,t)} \frac{\partial w^{(0)}(\xi,t)}{\partial \xi} \left(\dot{\mathbf{t}}^{(0)}(s,t) \cdot \mathbf{b}^{(0)}(s,t) \right) - 2\xi \frac{w^{(0)}(\xi,t)}{\sigma^{(0)}(s,t)} \frac{\partial K^{(0)}(s,t)}{\partial s} \tag{42}$$

$$
\begin{aligned}
Hs_{11}^{(2)} = {} & 2\xi K^{(0)}(s,t)\frac{\partial v_c^{(1)}(\xi,t)}{\partial \xi} + v^{(0)}(\xi,t)K^{(1)}(s,t) + 2\frac{\xi}{v^{(0)}(\xi,t)}\frac{\partial w^{(0)}(\xi,t)}{\partial \xi}\left(\dot{\mathbf{t}}^{(0)}(s,t)\cdot\mathbf{n}^{(0)}(s,t)\right) \\
& +6K^{(0)}(s,t)v_c^{(1)}(\xi,t) + 2\xi\frac{\partial v^{(0)}(\xi,t)}{\partial \xi}K^{(1)}(s,t) + 2\xi w^{(0)}(\xi,t)K^{(0)}(s,t)T^{(0)}(s,t) \\
& +2\frac{\xi K^{(0)}(s,t)v_c^{(1)}(\xi,t)}{v^{(0)}(\xi,t)}\frac{\partial v^{(0)}(\xi,t)}{\partial \xi} + 2\frac{\xi K^{(0)}(s,t)w_c^{(1)}(\xi,t)}{v^{(0)}(\xi,t)}\frac{\partial w^{(0)}(\xi,t)}{\partial \xi} \\
& +2\frac{\xi K^{(1)}(s,t)w^{(0)}(\xi,t)}{v^{(0)}(\xi,t)}\frac{\partial w^{(0)}(\xi,t)}{\partial \xi} + 2\frac{\xi K^{(0)}(s,t)w^{(0)}(\xi,t)}{v^{(0)}(\xi,t)}\frac{\partial w_c^{(1)}(\xi,t)}{\partial \xi}
\end{aligned}
\tag{43}
$$

This equation (39) is the first order equation of motion of the central line. Fukumoto and Miyazaki [4] (page 373) had written this equation without $K^{(1)}$, $\mathbf{E}_1$ and terms with the axial flux m.

V. CONCLUSION

A closed and complete system of equations (36,38,39) for the first order axisymmetric part $v_c^{(1)}, w_c^{(1)}$ of the velocity field and for the first order central line : $\mathbf{X}^{(1)}$ of a slender non circular vortex ring has been given.
It would be interesting to find a simple case where a solution to these equations can be found and to compare these equations with those of a circular vortex ring [5].

VI. APPENDIX

In this appendix, terms that appears in formulas of section 2 are given.

$m = \pi \int_0^\infty \omega_2 \overline{r}^2 d\overline{r} = 2\pi \int_0^\infty w^{(0)}\overline{r} d\overline{r}$

$\mathbf{I}_1 = \frac{m}{\pi}\mathbf{t}$

$\mathbf{I}_2 = \frac{1}{2\pi}Km\cos\varphi\mathbf{t}$

$\mathbf{I}_3 = m\left[(\frac{1}{8\pi}K^2 - \frac{1}{4\pi}T^2)\mathbf{t} - \frac{1}{4\pi}\left(KT\sin\varphi + 3K_s\cos\varphi\right)\mathbf{e}_r - \frac{1}{4\pi}\left(KT\cos\varphi - 3K_s\sin\varphi\right)\mathbf{e}_\theta\right]$

$$
\begin{aligned}
\mathbf{I}_4 = \frac{m}{4\pi}\Big\{ & \left(\left[-\tfrac{5}{6} + \ln S\right]KT\sin\varphi + \left[-3 - \tfrac{5}{6} + 3\ln S\right]K_s\cos\varphi\right)\mathbf{e}_r \\
& + \left(\left[-\tfrac{5}{6} + \ln S\right]KT\cos\varphi + \left[4 - \tfrac{1}{6} - 3\ln S\right]K_s\sin\varphi\right)\mathbf{e}_\theta \\
& + \left(\left(\left[-\tfrac{1}{2}\ln(2) + \tfrac{5}{16} - \tfrac{8}{S^2} + \tfrac{1}{4}\cos 2\varphi\right]K^2 + \left[\ln(2) - \tfrac{3}{2}\right]T^2\right)\mathbf{t}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}_1(a) = \frac{m}{4\pi}\Bigg\{ & \int_{-S/2}^{S/2} \frac{K(a+\overline{a})\mathbf{b}(a+\overline{a})\times(\mathbf{X}(a+\overline{a})-\mathbf{X}(a))+2T(a+\overline{a})}{\left|\mathbf{X}(a+\overline{a})-\mathbf{X}(a)\right|^3} \\
& +3\ \frac{\left[\mathbf{n}(a+\overline{a})\cdot(\mathbf{X}(a+\overline{a})-\mathbf{X}(a))\right]\left(\mathbf{b}(a+\overline{a})\times(\mathbf{X}(a+\overline{a})-\mathbf{X}(a))\right)}{\left|\mathbf{X}(a+\overline{a})-\mathbf{X}(a)\right|^3} \\
& -3\ \frac{\left[\mathbf{b}(a+\overline{a})\cdot(\mathbf{X}(a+\overline{a})-\mathbf{X}(a))\right]\left(\mathbf{n}(a+\overline{a})\times(\mathbf{X}(a+\overline{a})-\mathbf{X}(a))\right)}{\left|\mathbf{X}(a+\overline{a})-\mathbf{X}(a)\right|^3}
\end{aligned}
$$

$$+\left(-2\frac{1}{|\overline{a}|^3}+\frac{K(a)^2}{4|\overline{a}|}-\frac{T(a)^2}{2|\overline{a}|}\right)\mathbf{t}(a)-\frac{K(a)T(a)}{2|\overline{a}|}\mathbf{b}(a)-\frac{3}{2}\frac{K_a(a)}{|\overline{a}|}\mathbf{n}(a)d\overline{a}\Big\}$$

$$\mathbf{I}_5=-\frac{1}{4\pi}\left[(K_s\sin\varphi-KT\cos\varphi)\mathbf{t}+\frac{3}{4}K^2\sin(2\varphi)\mathbf{e}_r+\frac{3}{4}K^2\cos(2\varphi)\mathbf{e}_\theta\right]$$

$$\mathbf{I}_6=-\frac{1}{4\pi}\left[\begin{array}{l}(K_s\sin\varphi-KT\cos\varphi)\left[1-\ln S\right]\mathbf{t}+\left[1-\frac{3}{4}\ln S\right]K^2\sin(2\varphi)\mathbf{e}_r\\+\left[\frac{4}{S^2}-\frac{K^2}{24}+\left(\frac{5}{8}-\frac{3}{4}\ln S\right)K^2\cos(2\varphi)\right]\mathbf{e}_\theta\end{array}\right]$$

$$\mathbf{E}_2(\varphi,a)=\frac{1}{4\pi}\left(\mathbf{B}(\varphi,a)-3\mathbf{C}(\varphi,a)\right)$$

with :

$$\mathbf{A}(a)=\frac{1}{4\pi}\int_{-S/2}^{+S/2}\left[\frac{\mathbf{t}(a+\overline{a})\times(\mathbf{X}(a)-\mathbf{X}(a+\overline{a}))}{|\mathbf{X}(a)-\mathbf{X}(a+\overline{a})|^3}-\frac{K(a)\mathbf{b}(a)}{2|\overline{a}|}\right]d\overline{a}$$

$$\mathbf{B}(a)=\mathbf{e}_r(\varphi,a)\times\int_{-S/2}^{+S/2}\left[-\frac{\mathbf{t}(a+a')}{|\mathbf{X}(a)-\mathbf{X}(a+a')|^3}-f_b(a,a')\right]da'$$

$$\mathbf{C}(\varphi,a)=\int_{-S/2}^{+S/2}\left[\frac{\mathbf{e}_r(\varphi,a)\cdot(\mathbf{X}(a+a')-\mathbf{X}(a))}{|\mathbf{X}(a)-\mathbf{X}(a+a')|^5}\left[\mathbf{t}(a+a')\times(\mathbf{X}(a+a')-\mathbf{X}(a))\right]-f_c(a,a')\right]da'$$

$$f_b(a,a')=-\frac{1}{|a'|^3}\left[\mathbf{t}(a)+K(a)\mathbf{n}(a)a'+\frac{a'^2}{2}\left[K_a(a)\mathbf{n}(a)+K(a)T(a)\mathbf{b}(a)-\frac{3}{4}K^2(a)\mathbf{t}(a)\right]\right]$$

$$f_c(a,a')=-\frac{K^2(a)\mathbf{b}(a)\cos(\varphi)}{4|a'|}$$

where S is the length of the vortex ring.

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