THE COMPLETE FIRST ORDER EXPANSION OF A SLENDER VORTEX RING

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Abstract. Equations for the axisymmetric part of the velocity field and for the equation of motion of a *non circular* slender vortex ring are given at first order. This is the correction to the known leading order given by Callegari and Ting [2].

1. Definitions and Notations

The length scales of the vortex ring that are different from its thickness δ , for example : the radius of curvature, the ring length, are of the same order L with $\delta/L = O(\varepsilon) << 1$. The central curve is described parametricaly with the use of a function $\vec{\mathbf{X}} = \vec{\mathbf{X}}(s,t)$. A local curvilinear co-ordinate system (r,φ,s), with a frame $(\vec{\mathbf{r}},\vec{\theta},\vec{\tau})$, is introduced near this central curve [2]. There is an *outer problem* defined by the *outer limit* : $\varepsilon \to 0$ with r fixed, which describes the situation far from the central line and an *inner problem* defined by the *inner limit* : $\varepsilon \to 0$ with $\bar{r} = r/\varepsilon$ fixed, which describes the situation near the central line.

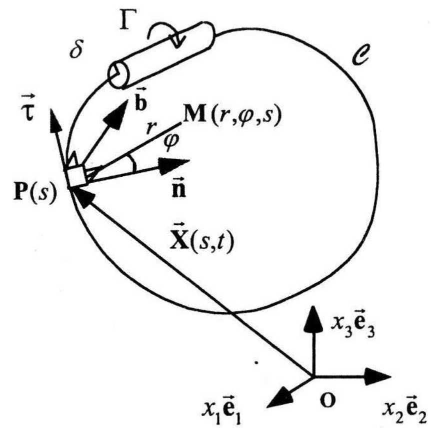

Figure 1 : The central curve and the local co-ordinates of the vortex ring.

E. Krause and K. Gersten (eds.), IUTAM Symposium on Dynamics of Slender Vortices, 45-54.

The change between Cartesian co-ordinates $\mathbf{M}(x_1,x_2,x_3)$ and local co-ordinates $\mathbf{M}(r,\varphi,s)$ satisfies :

$$\vec{\mathbf{x}} = \overrightarrow{\mathbf{OM}} = \vec{\mathbf{X}}(s,t) + r\vec{\mathbf{r}}(\varphi,s,t)$$

We have :

$$\sigma(s,t) = \left|\vec{\mathbf{X}}_s\right| \quad \vec{\mathbf{X}}_s = \sigma\vec{\tau} \quad \vec{\tau}_s = \sigma K\vec{\mathbf{n}}$$
$$\vec{\mathbf{n}}_s = \sigma(T\vec{\mathbf{b}} - K\vec{\tau}) \quad \vec{\mathbf{b}}_s = -\sigma T\vec{\mathbf{n}}$$
$$\vec{\mathbf{r}} = \vec{\mathbf{r}}(\varphi,s) = \vec{\mathbf{n}}(s)\cos\varphi + \vec{\mathbf{b}}(s)\sin\varphi$$
$$\vec{\theta} = \vec{\theta}(\varphi,s) = -\vec{\mathbf{n}}(s)\sin\varphi + \vec{\mathbf{b}}(s)\cos\varphi$$

where $| \; |$ is the usual norm of $\mathbf{R}^3$, T the local *torsion* of $\mathcal{C}$ and K the local *curvature* of $\mathcal{C}$. Notice that here and throughout this paper, the differentiation $\partial f/\partial x$ of a function f with respect to its variable x is denoted f_x ; $\times$ is the cross-product and $\bullet$ is the dot-product.

The small parameter ε is defined by $\varepsilon = \delta_0/L = \delta(t=0)/L$. Dimensionless variables :

$$r^* = r/L \quad \vec{\mathbf{X}}^* = \vec{\mathbf{X}}/L \quad \sigma^* = \sigma/L \quad K^* = LK$$
$$T^* = LT \quad S^* = S/L \quad \delta^* = \delta/L$$
$$t^* = t/(L^2/\Gamma) \quad \vec{\mathbf{v}}^* = \vec{\mathbf{v}}/(\Gamma/L) \quad \vec{\omega}^* = \vec{\omega}/(\Gamma/L^2)$$

are introduced, where S is the length of the ring and Γ is his circulation. Here, $\vec{\mathbf{v}}$ and $\vec{\omega}$ are respectively the velocity and the vorticity fields. From here on, all quantities are dimensionless and the asterisks are omitted.

The Reynolds number R_e is defined by $R_e = \Gamma/\nu$ where ν is the kinematic viscosity of the fluid. Let us define the number α such that $R_e^{-1/2} = \alpha\varepsilon$. Both inviscid : $\alpha = 0$ and viscous : $\alpha = O(1)$ vortex rings are studied.

The velocity is decomposed as follows :

$$\vec{\mathbf{v}}(r,\varphi,s,t,\varepsilon) = \dot{\vec{\mathbf{X}}}(s,t,\varepsilon) + \vec{\mathbf{V}}(r,\varphi,s,t,\varepsilon) \tag{1}$$

where
$$\vec{\mathbf{V}} = u\vec{\mathbf{r}} + v\vec{\theta} + w\vec{\tau} \tag{2}$$

and
$$\dot{\vec{\mathbf{X}}} = \frac{\partial\vec{\mathbf{X}}}{\partial t} \tag{3}$$

The following forms are chosen for the inner expansions of the velocity field :

$$\begin{aligned} u^{\mathrm{inn}} &= u^{(1)}(\bar{r},\varphi,s,t)+\ldots \\ v^{\mathrm{inn}} &= \varepsilon^{-1}v^{(0)}(\bar{r},s,t) + v^{(1)}(\bar{r},\varphi,s,t)+\ldots \\ w^{\mathrm{inn}} &= \varepsilon^{-1}w^{(0)}(\bar{r},s,t) + w^{(1)}(\bar{r},\varphi,s,t)+\ldots \end{aligned} \tag{4}$$

with an expression of the central curve of the form :

$$\vec{\mathbf{X}} = \vec{\mathbf{X}}^{(0)}(s,t) + \varepsilon\,\vec{\mathbf{X}}^{(1)}(s,t)+\ldots \tag{5}$$

2. Limit of $\vec{\mathbf{v}}^{\mathrm{inn}}$ at $\bar{r} \to \infty$ up to Order ε through Biot and Savart Law :

Let us have a vorticity field of the form :

$$\vec{\omega} = \frac{1}{\varepsilon^2}\vec{\omega}^{(0)}(\bar{r},\varphi,s) \tag{6}$$

The Biot and Savart law is given on local co-ordinates by the formula :

$$\vec{\mathbf{v}}(r,\varphi,s,t,\varepsilon) = \frac{1}{4\pi}\iiint \frac{\varepsilon^2\vec{\omega}(\bar{r}',\varphi',s',t,\varepsilon) \times \left[(\vec{\mathbf{X}}(s,t,\varepsilon) + r\,\vec{\mathbf{r}}(\varphi,s,t,\varepsilon)) - (\vec{\mathbf{X}}(s',t,\varepsilon) + \varepsilon\bar{r}'\,\vec{\mathbf{r}}')\right]}{\left|(\vec{\mathbf{X}}(s,t,\varepsilon) + r\,\vec{\mathbf{r}}(\varphi,s,t,\varepsilon)) - (\vec{\mathbf{X}}(s',t,\varepsilon) + \varepsilon\bar{r}'\,\vec{\mathbf{r}}')\right|^3} h_3'\,\bar{r}'\,d\bar{r}'\,d\varphi'\,ds' \tag{7}$$

where $h_3' = \sigma(s',t)\;(1 - K(s',t)\varepsilon\bar{r}'\cos(\varphi'))$.

Next, in this section, s will be an arc length parameter.

The outer expansion of velocity is :

$$\vec{\mathbf{v}}^{\mathrm{out}}(r,\varphi,s,\varepsilon) = \vec{\mathbf{v}}^{\mathrm{out}^{(0)}}(r,\varphi,s) + \varepsilon\,\vec{\mathbf{v}}^{\mathrm{out}^{(1)}}(r,\varphi,s) + O(\varepsilon^2).$$

If
$$\iint(\vec{\omega} - [\vec{\omega}\bullet\vec{\tau}]\vec{\tau}\,)\bar{r}d\bar{r}d\varphi = 0 \tag{8}$$

one obtains :

$$\vec{\mathbf{v}}^{\mathrm{out}(0)}(r,\varphi,s,\varepsilon) = \frac{1}{4\pi}\int_{\mathcal{C}} \frac{\vec{\tau}(s')\times(\vec{\mathbf{x}} - \vec{\mathbf{X}}(s'))}{\left|(\vec{\mathbf{x}} - \vec{\mathbf{X}}(s')\right|^3}ds' \tag{9a}$$

$$\begin{aligned}\vec{\mathbf{v}}^{\mathrm{out}(1)}(r,\varphi,s) = {}& \frac{1}{4\pi}\iiint \frac{\vec{\omega}^{(0)'} \times (\vec{\mathbf{x}} - \vec{\mathbf{X}}(s'))}{\left|\vec{\mathbf{x}} - \vec{\mathbf{X}}(s')\right|^3}\bar{r}'^2 K(s')\cos\varphi' d\bar{r}'\,d\varphi' ds' - \frac{1}{4\pi}\iiint \frac{\left(\vec{\mathbf{r}}' \times \vec{\omega}^{(0)'}\right)}{\left|\vec{\mathbf{x}} - \vec{\mathbf{X}}(s')\right|^3}\bar{r}'^2\,d\bar{r}'\,d\varphi' ds' \\ & - \frac{1}{4\pi}\iiint 3\frac{\left[\vec{\omega}^{(0)'} \times (\vec{\mathbf{x}} - \vec{\mathbf{X}}(s'))\right]\left(\vec{\mathbf{r}}' \bullet (\vec{\mathbf{x}} - \vec{\mathbf{X}}(s'))\right)}{\left|\vec{\mathbf{x}} - \vec{\mathbf{X}}(s')\right|^5}\bar{r}'^2\,d\bar{r}'\,d\varphi' ds'\end{aligned} \tag{9b}$$

with : $\quad \vec{\mathbf{x}} = \vec{\mathbf{X}}(s,t) + r\vec{\mathbf{r}}(\varphi,s,t)$.

Thus at leading order in outer co-ordinates, the velocity field exactly correspond to the Dirac delta distribution $\delta_{\mathcal{C}}\vec{\tau}$ on the central line.

In case $\vec{\omega}^{(0)} = \omega_2(\bar{r})\vec{\theta} + \omega_3(\bar{r})\vec{\tau}$, when $r = \varepsilon\,\bar{r}$ is put in $\vec{\mathbf{v}}^{\mathrm{out}}(r \to 0,\varphi,s)$, one obtains :

$$\begin{aligned}\vec{\mathbf{v}}^{\mathrm{inn}}(\bar{r} \to \infty,\varphi,s) = {}& \frac{1}{\varepsilon}\vec{\mathbf{v}}^{\mathrm{inn}(0)}(\bar{r} \to \infty,\varphi,s) + \ln\varepsilon\,\vec{\mathbf{v}}^{\mathrm{inn}(01)}(\bar{r} \to \infty,\varphi,s) + \vec{\mathbf{v}}^{\mathrm{inn}(1)}(\bar{r} \to \infty,\varphi,s) \\ & + \varepsilon\ln\varepsilon\,\vec{\mathbf{v}}^{\mathrm{inn}(12)}(\bar{r} \to \infty,\varphi,s) + \varepsilon\,\vec{\mathbf{v}}^{\mathrm{inn}(2)}(\bar{r} \to \infty,\varphi,s) + O(\varepsilon^2\ln\varepsilon)\end{aligned} \tag{10}$$

with :

$$\vec{\mathbf{v}}^{\,\mathrm{inn}(0)}(\bar{r}\to\infty,\varphi,s)=\frac{1}{2\pi}\frac{\vec{\theta}}{\bar{r}}+\frac{\vec{\mathbf{I}}_1}{\bar{r}^2}+O(\frac{1}{\bar{r}^3}) \qquad (11a)$$

$$\vec{\mathbf{v}}^{\,\mathrm{inn}(01)}(\bar{r}\to\infty,\varphi,s)=-\frac{K}{4\pi}\vec{\mathbf{b}} \qquad (11b)$$

$$\vec{\mathbf{v}}^{\,\mathrm{inn}(1)}(\bar{r}\to\infty,\varphi,s)=\frac{K}{4\pi}\left[\ln\frac{S}{\bar{r}}-1\right]\vec{\mathbf{b}}+\frac{K}{4\pi}\cos\varphi\vec{\theta}+\vec{\mathbf{A}}+\frac{\vec{\mathbf{I}}_2}{\bar{r}}+O(\frac{1}{\bar{r}^2}) \qquad (11c)$$

$$\vec{\mathbf{v}}^{\,\mathrm{inn}(12)}(\bar{r}\to\infty,\varphi,s)=\vec{\mathbf{I}}_3+\vec{\mathbf{I}}_5\bar{r} \qquad (11d)$$

$$\vec{\mathbf{v}}^{\,\mathrm{inn}(2)}(\bar{r}\to\infty,\varphi,s)=(\vec{\mathbf{I}}_3+\vec{\mathbf{I}}_5\bar{r})\ln\bar{r}+\left(\vec{\mathbf{I}}_6+\vec{\mathbf{E}}_2(\varphi,s)\right)\bar{r}+\vec{\mathbf{I}}_4+\vec{\mathbf{E}}_1(s) \qquad (11e)$$

$$\vec{\mathbf{E}}_2(\varphi,s)=\frac{1}{4\pi}\left(\vec{\mathbf{B}}(\varphi,s)-3\vec{\mathbf{C}}(\varphi,s)\right) \qquad (11f)$$

where expressions of $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, $\vec{\mathbf{C}}$, $\vec{\mathbf{E}}_i$, $\vec{\mathbf{I}}_i$ (i=1...6) are given in Appendix.

This expression (10) can be compared with that of Fukumoto and Miyazaki [4] (page 373) and Callegari et Ting [2] (page 173). It is the same all but here $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, $\vec{\mathbf{C}}$ and order ε are given. Besides here the derivation was performed in an algorithmic way with formal calculus (Maple) and with the matched asymptotic expansion of singular integral method following François [3] or Bender and Orszag [1].

Let us notice that the same result would have be obtained if $\bar{r}\to\infty$ were put in the inner expansion of Biot and Savart law [6].

This result will be used in the following when the asymptotic matching will be performed.

3. Results at Order 0

Callegari and Ting [2] considered the case where $v^{(0)}, w^{(0)}$ are independent of s so that some compatibility conditions are satisfied. They deduced the following equations for $v^{(0)}, w^{(0)}$ from Navier Stokes second order equations :

$$\bar{r}\frac{\partial v^{(0)}(\bar{r},t)}{\partial t}-\alpha^2\frac{\partial v^{(0)}(\bar{r},t)}{\partial\bar{r}}+\frac{\alpha^2}{\bar{r}}v^{(0)}(\bar{r},t)-\alpha^2\bar{r}\frac{\partial^2 v^{(0)}(\bar{r},t)}{\partial\bar{r}^2}-\frac{1}{2}\bar{r}\frac{\partial\,\bar{r}v^{(0)}(\bar{r},t)}{\partial\bar{r}}\frac{\dot{S}^{(0)}}{S^{(0)}}=0 \qquad (12a)$$

$$\bar{r}\frac{\partial w^{(0)}(\bar{r},t)}{\partial t}-\alpha^2\frac{\partial w^{(0)}(\bar{r},t)}{\partial\bar{r}}-\alpha^2\bar{r}\frac{\partial^2 w^{(0)}(\bar{r},t)}{\partial\bar{r}^2}-\frac{1}{2}\bar{r}^4\left(\frac{w^{(0)}(\bar{r},t)}{\bar{r}^2}\right)_{\bar{r}}\frac{\dot{S}^{(0)}}{S^{(0)}}=0 \qquad (12b)$$

where $S^{(0)}$ is the length of the ring.

Through matching, they found the following equation for $\vec{\mathbf{X}}^{(0)}(s,t)$:

$$\dot{\vec{\mathbf{X}}}^{(0)}-(\dot{\vec{\mathbf{X}}}^{(0)}\bullet\vec{\tau})\vec{\tau}=(\frac{K^{(0)}}{4\pi}\left[\ln\frac{S^{(0)}}{\varepsilon}-1\right]+K^{(0)}C^*)\vec{\mathbf{b}}+\vec{\mathbf{A}}-(\vec{\mathbf{A}}\bullet\vec{\tau})\vec{\tau} \qquad (13a)$$

where

$$C^*(t)=\frac{1}{4\pi}\left\{+\frac{1}{2}+\lim_{\bar{r}\to\infty}\left(4\pi^2\int_0^{\bar{r}}\xi\left(v^{(0)}\right)^2 d\xi-\ln(\bar{r})\right)-8\pi^2\int_0^{\infty}\xi(w^{(0)})^2\,d\xi\right\} \qquad (13b)$$

$$\lambda(s,\bar{s},t)=\int_s^{s+\bar{s}}\sigma^{(0)}(s^*,t)ds^* \qquad (13c)$$

$$\vec{\mathbf{A}}(s,t)=\frac{1}{4\pi}\int_{-\pi}^{+\pi}\left[-\sigma^{(0)}(s+\bar{s},t)\frac{\vec{\tau}^{(0)}(s+\bar{s},t)\times(\vec{\mathbf{X}}^{(0)}(s+\bar{s},t)-\vec{\mathbf{X}}^{(0)}(s,t))}{\left|\vec{\mathbf{X}}^{(0)}(s+\bar{s},t)-\vec{\mathbf{X}}^{(0)}(s,t)\right|^3}-\frac{K^{(0)}(s,t)}{2}\frac{\vec{\mathbf{b}}^{(0)}(s,t)\sigma^{(0)}(s+\bar{s},t)}{\left|\lambda^{(0)}(s,\bar{s},t)\right|}\right]d\bar{s} \qquad (13d)$$

4. Results at Order 1

In the same way that first order Navier Stokes equations give compatibility equations for $v^{(0)}, w^{(0)}$, second order Navier Stokes equations give compatibility equations for the axisymmetric part $v_c^{(1)}, w_c^{(1)}$ of $v^{(1)}, w^{(1)}$. These equations are automatically satisfied if $v_c^{(1)}$ is assumed to be independent of s and if $w_c^{(1)}$ is such that :

$$\frac{\partial w_c^{(1)}(\bar{r},s,t)}{\partial s}=-\dot{\sigma}^{(0)}+a(s,t)\sigma^{(0)} \qquad (14a)$$

$$a(s,t)=\dot{S}^{(0)}/S^{(0)} \qquad (14b)$$

We write :

$$w_c^{(1)}(\bar{r},s,t)=w_{cc}^{(1)}(s,t)+w_c^{(1)}(\bar{r},t) \qquad (14c)$$

Equations for $v_c^{(1)}, w_c^{(1)}$ can be extracted from third order Navier Stokes equations. We did this with the use of symbolic calculus (on maple) in the following way : we obtained third order Navier Stokes equations, then we carried out the φ-average and s-average of the axial and circumferential components of the third order momentum equations using the third order continuity equation to eliminate $u_c^{(2)}$. Lots of terms vanished and we found equations for $v_c^{(1)}, w_c^{(1)}$. Note that Fukumoto and Miyazaki ([4] page 378) postulated $v_c^{(1)}=0$, kept $w_c^{(1)}$ dependent of s, and did not have equation for $w_c^{(1)}$. The following equations are obtained :

$$S^{(0)}\left(\frac{\partial v_c^{(1)}}{\partial t}-\alpha^2\left[\frac{1}{\bar{r}}(\bar{r}v_c^{(1)})_{\bar{r}}\right]_{\bar{r}}\right)-\frac{1}{2}\dot{S}^{(0)}\left(\bar{r}v_c^{(1)}\right)_{\bar{r}}=\frac{\bar{r}}{2}\left(\dot{S}^{(1)}-S^{(1)}\frac{\dot{S}^{(0)}}{S^{(0)}}\right)\zeta^{(0)} \tag{14d}$$

where $S^{(1)}=\int_0^{2\pi}\sigma^{(1)}ds$.

$$\begin{aligned}S^{(0)}\left(\frac{\partial w_c^{(1)}}{\partial t}-\alpha^2\left[\frac{1}{\bar{r}}(\bar{r}w_{c\,\bar{r}}^{(1)})_{\bar{r}}\right]_{\bar{r}}\right)-\frac{1}{2}\dot{S}^{(0)}\bar{r}^3\left(\frac{w_c^{(1)}}{\bar{r}^2}\right)_{\bar{r}} &= \frac{\bar{r}^3}{2}(w^{(0)}/\bar{r}^2)_{\bar{r}}\left(\dot{S}^{(1)}-S^{(1)}\frac{\dot{S}^{(0)}}{S^{(0)}}\right)\\ &+\frac{1}{4\pi}\left(\ln(\frac{S}{\varepsilon})+C-4\pi\bar{r}v^{(0)}\right)\int_0^{2\pi}K^{(0)}\vec{\mathbf{A}}_s(s,t)\bullet\vec{\mathbf{b}}^{(0)}ds-\int_0^{2\pi}\sigma^{(0)}\vec{\mathbf{A}}(s,t)\bullet\vec{\tau}^{(0)}ds\\ &-\int_0^{2\pi}\sigma^{(0)}\frac{\partial w_{cc}^{(1)}(s,t)}{\partial t}ds-\frac{\dot{S}^{(0)}}{S^{(0)}}\int_0^{2\pi}\sigma^{(0)}w_{cc}^{(1)}(s,t)ds\end{aligned} \tag{14e}$$

The left hand side of these equations are the same for $v_c^{(1)}(\bar{r},t)$ and $w_c^{(1)}(\bar{r},t)$ than for $v^{(0)}(\bar{r},t)$ and $w^{(0)}(\bar{r},t)$. We may notice that even if initially $w_c^{(1)}(\bar{r},0)=0$, the right hand side terms will induce $w_c^{(1)}(\bar{r},0)\neq 0$ when $t\neq 0$.

These equations for $v_c^{(1)}, w_c^{(1)}$ are linked to $\vec{\mathbf{X}}^{(1)}(s,t)$, so an equation for $\vec{\mathbf{X}}^{(1)}$ is needed to have a closed system of equations for the first order solutions $v_c^{(1)}, w_c^{(1)}$ and $\vec{\mathbf{X}}^{(1)}$. The best attempt to find this equation is by Fukumoto and Miyazaki ([4] page 382), where contribution from Navier Stokes equations up to order ε^1 have been found in order to performed the asymptotic matching. We may note that in their expression the term due to first order curvature $K^{(1)}$ is missing. Moreover, their expression for $\vec{\mathbf{X}}^{(1)}$ is not complete, as local and non-local (named $\vec{\mathbf{Q}}$ in [4]) contribution from Biot and Savart integral are given only up to order ε^0 in Fukumoto and Miyazaki ([4] page 373) and Callegari and Ting ([2] page 173), while order ε^1 is obviously needed to obtain complete and correct equation for $\vec{\mathbf{X}}^{(1)}$. The complete expression, up to ε order, was performed in section 2. The matching is then done and lacking terms in equation of $\vec{\mathbf{X}}^{(1)}$ are found :

$$\begin{aligned}\dot{\vec{\mathbf{X}}}^{(1)}-(\dot{\vec{\mathbf{X}}}^{(1)}\bullet\vec{\tau})\vec{\tau} &=\left\{\overset{*}{C}_2-\frac{1}{8\pi}K^{(1)}(s,t)+\frac{1}{4\pi}K^{(1)}(s,t)\ln\varepsilon-\frac{m}{4\pi}K_s\left[3\ln\varepsilon+3+\frac{5}{6}-3\ln S\right]\right\}\vec{\mathbf{n}}\\ &+\left\{\overset{*}{C}_1+\frac{m}{4\pi}KT\left[\ln\varepsilon+\frac{5}{6}-\ln S\right]\right\}\vec{\mathbf{b}}+\vec{\mathbf{E}}_1-(\vec{\mathbf{E}}_1\bullet\vec{\tau})\vec{\tau}\end{aligned} \tag{15a}$$

$$\overset{*}{C}_1=\pi\int_0^{\infty}\xi v^{(0)}(\xi,t)Hs_{12}^{(2)}(\xi,t)d\xi \tag{15b}$$

$$\overset{*}{C}_2=\pi\lim_{\bar{r}\to\infty}\left(\int_0^{\bar{r}}\xi v^{(0)}(\xi,t)Hs_{11}^{(2)}(\xi,t)d\xi-\frac{1}{4}\frac{K^{(1)}(s,t)}{\pi^2}\ln\bar{r}\right) \tag{15c}$$

$$Hs_{12}^{(2)}=2\frac{\xi}{v^{(0)}(\xi,t)}\frac{\partial w^{(0)}(\xi,t)}{\partial\xi}\left(\dot{\vec{\tau}}^{(0)}(s,t)\bullet\vec{\mathbf{b}}^{(0)}(s,t)\right)-2\xi\frac{w^{(0)}(\xi,t)}{\sigma^{(0)}(s,t)}\frac{\partial K^{(0)}(s,t)}{\partial s} \tag{15d}$$

$$\begin{aligned}Hs_{11}^{(2)} &=2\xi K^{(0)}(s,t)\frac{\partial v_c^{(1)}(\xi,t)}{\partial\xi}+v^{(0)}(\xi,t)K^{(1)}(s,t)+2\frac{\xi}{v^{(0)}(\xi,t)}\frac{\partial w^{(0)}(\xi,t)}{\partial\xi}\left(\dot{\vec{\tau}}^{(0)}(s,t)\bullet\vec{\mathbf{n}}^{(0)}(s,t)\right)\\ &+6K^{(0)}(s,t)v_c^{(1)}(\xi,t)+2\xi\frac{\partial v^{(0)}(\xi,t)}{\partial\xi}K^{(1)}(s,t)+2\xi w^{(0)}(\xi,t)K^{(0)}(s,t)T^{(0)}(s,t)\\ &+2\frac{\xi K^{(0)}(s,t)v_c^{(1)}(\xi,t)}{v^{(0)}(\xi,t)}\frac{\partial v^{(0)}(\xi,t)}{\partial\xi}+2\frac{\xi K^{(0)}(s,t)w_c^{(1)}(\xi,t)}{v^{(0)}(\xi,t)}\frac{\partial w^{(0)}(\xi,t)}{\partial\xi}\\ &+2\frac{\xi K^{(1)}(s,t)w^{(0)}(\xi,t)}{v^{(0)}(\xi,t)}\frac{\partial w^{(0)}(\xi,t)}{\partial\xi}+2\frac{\xi K^{(0)}(s,t)w^{(0)}(\xi,t)}{v^{(0)}(\xi,t)}\frac{\partial w_c^{(1)}(\xi,t)}{\partial\xi}\end{aligned} \tag{15e}$$

This equation (15a) is the first order equation of motion of the central line. Fukumoto and Miyazaki [4] (page 373) had written this equation without $K^{(1)}$, $\vec{\mathbf{E}}_1$ and terms with the axial flux m. .

5. Conclusion :

A closed and complete system of equations (14d,14e,15a) for the first order axisymmetric part $v_c^{(1)}, w_c^{(1)}$ of the velocity field and for the first order central line : $\vec{\mathbf{X}}^{(1)}$ of a slender *non circular* vortex ring has been given.

It would be interesting to find a simple case where a solution to these equations can be found and to compare these equations with those of a circular vortex ring [5].

6. Appendix :

In this appendix, terms that appears in formulas of section 2 are given.

$$m=\pi\int_0^\infty \omega_2 \bar{r}^{-2} d\bar{r} = 2\pi\int_0^\infty w^{(0)}\bar{r}d\bar{r} \qquad \vec{\mathbf{I}}_1 = \frac{m}{\pi}\vec{\tau} \qquad \vec{\mathbf{I}}_2 = \frac{1}{2\pi}Km\cos\varphi\vec{\tau}$$

$$\vec{\mathbf{I}}_3 = m\left[(\frac{1}{8\pi}K^2 - \frac{1}{4\pi}T^2)\vec{\tau} - \frac{1}{4\pi}\left(KT\sin\varphi + 3K_s\cos\varphi\right)\vec{\mathbf{r}} - \frac{1}{4\pi}\left(KT\cos\varphi - 3K_s\sin\varphi\right)\vec{\theta}\right]$$

$$\begin{aligned}\vec{\mathbf{I}}_4 = \frac{m}{4\pi}\Bigg\{ &\left(\left[-\frac{5}{6} + \ln S\right]KT\sin\varphi + \left[-3 - \frac{5}{6} + 3\ln S\right]K_s\cos\varphi\right)\vec{\mathbf{r}} \\ &+ \left(\left[-\frac{5}{6} + \ln S\right]KT\cos\varphi + \left[4 - \frac{1}{6} - 3\ln S\right]K_s\sin\varphi\right)\vec{\theta} \\ &+ (\left(\left[-\frac{1}{2}\ln(2) + \frac{5}{16} - \frac{8}{S^2} + \frac{1}{4}\cos 2\varphi\right]K^2 + \left[\ln(2) - \frac{3}{2}\right]T^2\right)\vec{\tau}\Bigg\}\end{aligned}$$

$$\begin{aligned}\vec{\mathbf{E}}_1(a) = \frac{m}{4\pi}\Bigg\{ &\int_{-S/2}^{S/2} \frac{K(a+\bar{a})\vec{\mathbf{b}}(a+\bar{a}) \times (\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a)) + 2T(a+\bar{a})}{\left|\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a)\right|^3} \\ &+ 3\frac{\left[\vec{\mathbf{n}}(a+\bar{a}) \bullet (\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a))\right]\left(\vec{\mathbf{b}}(a+\bar{a}) \times (\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a))\right)}{\left|\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a)\right|^3} \\ &- 3\frac{\left[\vec{\mathbf{b}}(a+\bar{a}) \bullet (\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a))\right]\left(\vec{\mathbf{n}}(a+\bar{a}) \times (\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a))\right)}{\left|\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a)\right|^3} \\ &+ \left(-2\frac{1}{\left|\bar{a}\right|^3} + \frac{K(a)^2}{4\left|\bar{a}\right|} - \frac{T(a)^2}{2\left|\bar{a}\right|}\right)\vec{\tau}(a) - \frac{K(a)T(a)}{2\left|\bar{a}\right|}\vec{\mathbf{b}}(a) - \frac{3}{2}\frac{K_a(a)}{\left|\bar{a}\right|}\vec{\mathbf{n}}(a) \qquad d\bar{a}\Bigg\}\end{aligned}$$

$$\vec{\mathbf{I}}_5 = -\frac{1}{4\pi}\left[(K_s\sin\varphi - KT\cos\varphi)\vec{\tau} + \frac{3}{4}K^2\sin(2\varphi)\vec{\mathbf{r}} + \frac{3}{4}K^2\cos(2\varphi)\vec{\theta}\right]$$

$$\vec{\mathbf{I}}_6 = -\frac{1}{4\pi}\left[\begin{aligned}&(K_s\sin\varphi - KT\cos\varphi)[1 - \ln S]\vec{\tau} + \left[1 - \frac{3}{4}\ln S\right]K^2\sin(2\varphi)\vec{\mathbf{r}} \\ &+ \left[\frac{4}{S^2} - \frac{K^2}{24} + \left(\frac{5}{8} - \frac{3}{4}\ln S\right)K^2\cos(2\varphi)\right]\vec{\theta}\end{aligned}\right]$$

$$\vec{\mathbf{E}}_2(\varphi,a) = \frac{1}{4\pi}\left(\vec{\mathbf{B}}(\varphi,a) - 3\vec{\mathbf{C}}(\varphi,a)\right)$$

with :

$$\vec{\mathbf{A}}(a) = \frac{1}{4\pi}\int_{-S/2}^{+S/2} \frac{\vec{\tau}(a+\bar{a}) \times (\vec{\mathbf{X}}(a) - \vec{\mathbf{X}}(a+\bar{a}))}{\left|\vec{\mathbf{X}}(a) - \vec{\mathbf{X}}(a+\bar{a})\right|^3} - \frac{K(a)\vec{\mathbf{b}}(a)}{2\left|\bar{a}\right|}d\bar{a}$$

$$\vec{\mathbf{B}}(\varphi,a) = \vec{\mathbf{r}}(\varphi,a) \times \int_{-S/2}^{+S/2}\left[\begin{aligned}&-\frac{\vec{\tau}(a+\bar{a})}{\left|\vec{\mathbf{X}}(a) - \vec{\mathbf{X}}(a+\bar{a})\right|^3} \\ &+ \frac{1}{\left|\bar{a}\right|^3}(\vec{\tau}(a) + K(a)\vec{\mathbf{n}}(a)\bar{a} + (K_a(a)\vec{\mathbf{n}}(a) + K(a)T(a)\vec{\mathbf{b}}(a) \\ &- \frac{3}{4}K^2(a)\vec{\tau}(a))\frac{\bar{a}^2}{2}\end{aligned}\right]d\bar{a}$$

$$\vec{\mathbf{C}}(\varphi,a) = \int_{-S/2}^{+S/2}\left[\frac{\vec{\mathbf{r}}(\varphi,a) \bullet (\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a))}{\left|\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a)\right|^5}\left[\vec{\tau}(a+\bar{a}) \times (\vec{\mathbf{X}}(a+\bar{a}) - \vec{\mathbf{X}}(a))\right] + \frac{K^2(a)}{4}\frac{\vec{\mathbf{b}}(a)\cos\varphi}{\left|\bar{a}\right|}\right]d\bar{a}$$

where S is the length of the vortex ring.

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