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Motion and oscillations of a circular perturbed vortex ring

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Abstract. The long bending distortions of the central line of a perturbed circular slender vortex ring with an axial velocity component are studied with the equation of motion of Callegari and Ting [1] rather than with the equation due to a cut-off method as in Widnall and Sullivan [2]. The link between the evolution of the perturbation and the inner structure of the vortex is then shown and the study of the perturbations of small amplitudes gives an analytical expression of the period of oscillation for a circular perturbed inviscid vortex. A numerical code that solves the full nonlinear filament evolution equation has been developed and validated by the results of the linear analysis. In the small amplitude limit, the numerical simulations and the analytical approach are in excellent agreement both for an inviscid and a viscous vortex. The evolution of a perturbation of finite amplitude is also given. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

vortex ring / oscillation / equation of motion / numeric / perturbation

Mouvement et oscillations d'un anneau tourbillon circulaire perturbé

Résumé. *Dans cette note, les distortions de la fibre centrale d'un anneau tourbillon circulaire perturbé sont étudiées avec l'équation du mouvement de Callegari et Ting [1], plutôt qu'avec une équation due à une méthode de coupure comme dans Widnall et Sullivan [2]. Ceci permet de faire apparaître le lien entre l'évolution de la perturbation et la structure interne du tourbillon. Une étude analytique des perturbations de faibles amplitudes donne une expression analytique de la période d'oscillation pour un anneau circulaire perturbé inviscide. Un code numérique qui résoud l'équation non linéaire d'évolution du filament a été développé et validé par les résultats de l'étude linéaire. Dans la limite des amplitudes faibles, les simulations numériques et l'approche analytique sont en excellent accord, que ce soit pour un fluide parfait ou non. L'évolution d'une perturbation d'amplitude finie est également donnée. © 2000 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS*

anneau tourbillon / oscillation / équation du mouvement / numérique / perturbation

Version française abrégée

1. Introduction

Après avoir défini un anneau tourbillon de faible épaisseur et introduit la fonction $\boldsymbol{X} = \boldsymbol{X}(s, t)$, qui repère sa fibre centrale, l'équation (1) de Callegari et Ting [1] d'évolution de cette courbe est donnée.

Note présentée par René MOREAU.

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C'est une équation aux dérivées partielles du temps t et du paramètre s sur la courbe. On montre alors que la courbe circulaire qui se translate sans se déformer est une solution de cette équation. Afin d'étudier les perturbations de cette solution, l'angle θ des coordonnées cylindriques associées à la géométrie axisymétrique de cette solution est choisi pour repérer l'abscisse s sur la courbe. On se propose de se servir de l'équation (1) pour étudier l'évolution de ces perturbations en précisant le lien avec la structure du filament et ainsi étendre l'étude de Widnall et Sullivan [2]. De plus, à l'aide de (1), une étude visqueuse et d'amplitude finie peut-être envisagée.

2. Étude analytique de la perturbation dans la limite des faibles amplitudes

L'équation d'évolution des amplitudes ρ et ξ de la perturbation est donnée par le système linéaire (2) dans le cas des amplitudes faibles. On aboutit à une formule analytique (3) de la période d'oscillation pour un anneau inviscide et à une résolution numérique par la méthode de Runge–Kutta de ce système dans le cas visqueux.

3. Étude numérique et comparaison avec l'étude analytique

L'équation (1) a été résolue numériquement. Pour les faibles amplitudes, on retrouve bien les résultats de l'étude analytique précédente que ce soit pour un anneau inviscide (*figure 2*) ou visqueux (*figure 3*). Une évolution à amplitude finie est alors décrite (*figure 4*).

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1. Introduction

In this introduction, after having given the definition of a slender vortex ring and the equation of motion of Callegari and Ting [1] (hereinafter referred to as CT), we show that the circular vortex is a solution to this equation and we explain how the study of the perturbations of this solution with the equation of CT allows to extend the work of Widnall and Sullivan [2] (hereinafter referred to as WS).

A *slender vortex ring* of circulation Γ is a field of vorticity which is non-zero only in the neighbourhood of a three-dimensional curve $\mathcal{C}$, called the *central line*. This curve is described parametrically by a function $\boldsymbol{X} = \boldsymbol{X}(s,t)$ which denotes a point on the curve as a function of the parameter s, with $s \in [-\pi, \pi[$, and the time t. For each point on the curve $\mathcal{C}$, the Frenet frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ is defined with respectively the tangent, normal and binormal vectors. The thickness δ of the ring is of order l and the other length scales, for example the local *curvature* K or the length S of $\mathcal{C}$ are of the same order L. Since the vortex is slender, a small parameter $\varepsilon \ll 1$ is defined as the ratio l/L. The Reynolds number $R_e = \Gamma/\nu$, where ν is the kinematic viscosity, is linked to ε by $R_e^{-1/2} = \alpha\varepsilon$, where the *viscous number* $\alpha = \mathrm{O}(1)$ is defined; the inviscid vortex ring is obtained in the limit $\alpha = 0$. The velocity field is non-dimensionalized using Γ/L, all lengths using L, and the time using L^2/Γ.

The equation of motion of a *non-circular* slender vortex ring has been derived by CT from the Navier–Stokes equations using a matched asymptotic expansion in ε. At leading order the following equation was obtained:

$$\partial\boldsymbol{X}/\partial t - [\partial\boldsymbol{X}/\partial t \cdot \boldsymbol{t}]\boldsymbol{t}(s,t) = \boldsymbol{Q}(s,t) + K(s,t)\big[-\ln\varepsilon + \ln S(t) - 1 + C_v(t) + C_w(t)\big]\boldsymbol{b}(s,t)/4\pi \quad (1)$$

where $\boldsymbol{Q}(s,t)$ is an integral given by $\boldsymbol{Q}(s,t) = \boldsymbol{A}(s,t) - [\boldsymbol{A}(s,t)\cdot\boldsymbol{t}(s,t)]\boldsymbol{t}(s,t)$ with:

$$\boldsymbol{A}(s,t) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \sigma(s+s',t)\left[\frac{\boldsymbol{t}(s+s',t)\times(\boldsymbol{X}(s,t)-\boldsymbol{X}(s+s',t))}{|\boldsymbol{X}(s,t)-\boldsymbol{X}(s+s',t)|^3} - \frac{K(s,t)\boldsymbol{b}(s,t)}{2|\lambda(s,s',t)|}\right]\mathrm{d}s'$$

Here $\sigma(s,t) = |\partial\boldsymbol{X}/\partial s|$ and $\lambda(s,s',t) = \int_s^{s+s'}\sigma(s^*,t)\,\mathrm{d}s^*$. In equation (1) $C_v(t)$ and $C_w(t)$ are known functions (see [1]) depending on the inner structure of the vortex at leading order in ε. This equation holds for a vortex ring with an axisymmetric core structure at leading order and without axial variations. It does not take into account the fast core oscillations of small amplitude ε^2 that may occur on a time t/ε^2. Any axial variation would evolve on a time t/ε and would require a two-time analysis with this new small time scale.

The leading order velocity field in the core is of order $1/\varepsilon$. It is given by the orthoradial velocity $v^{(0)}$ and the axial velocity $w^{(0)}$, that are only function of the radial stretched distance $\bar{r} = r/\varepsilon$ to the filament, where r is the dimensionless distance (using L as a typical length scale). Equations of evolutions of this leading order field have been extracted [1] from the Navier–Stokes equations and are coupled to the equation of motion of the filament by the stretching rate $\dot{S}/S$, where $\dot{S}$ is the time derivative of the length of the filament. A special and interesting solution of these core equations is the *similar* vortex ring [1]:

$$v^{(0)} = \frac{1}{2\pi\bar{r}}\left[1 - \mathrm{e}^{-(\bar{r}/\bar{\delta})^2}\right], \qquad w^{(0)} = \frac{m_0}{\pi\bar{\delta}^2}\left(\frac{S_0}{S}\right)\mathrm{e}^{-(\bar{r}/\bar{\delta})^2}, \qquad \tau_\alpha = 4\alpha^2\int_0^t S(t')/S_0\,\mathrm{d}t'$$

where m_0 is the initial axial flux of the ring, $\bar{\delta} = \delta/\varepsilon = (S_0/S)^{1/2}(1+\tau_\alpha)^{1/2}$ is the stretched radius and τ_α the diffusion time. Here the typical short length scale l is chosen to be the initial thickness δ_0. For such a vortex, the inner functions are given by $C_v(t) = [1+\gamma-\ln 2]/2 - \ln(\bar{\delta})$ and $C_w(t) = -2(S_0/S)^4(m_0/\bar{\delta})^2$, where γ is the Euler number. The effect of the diffusion is easy seen in these expressions through the coefficient τ_α and the one of the stretching through the ratio S_0/S. It is not a very restrictive assumption to study a similar vortex, for it has been shown [1] that the amplitude of any non-similar part decays as $(1+\tau_\alpha)^{-n}$, where $n \geqslant 1$ is an integer.

The *circular vortex ring* of radius R_0 is an exact solution of (1). This vortex remains circular ($S(t) = S_0$) and moves with the velocity $\partial\boldsymbol{X}/\partial t = \widetilde{V}_\alpha \boldsymbol{b}/4\pi$, where $\widetilde{V}_\alpha = \ln(8/\varepsilon) + C_v(t) - 1 + C_w(t)$. Moreover for a similar vortex, $C_w(t)$ and $\bar{\delta}(t)$ are then given by $C_w(t) = -2(m_0/\bar{\delta})^2$ and $\bar{\delta}(t) = \sqrt{1+4\alpha^2 t}$. Here and in the following, the typical length scale L is chosen to be the initial radius R_0.

Let us now consider *perturbations* of the central line of this circular vortex ring. Since the basic flow $\boldsymbol{X}_b$ is axisymmetric, the *central line* of the perturbed circular vortex ring is described parametrically with the angle θ of cylindrical coordinates associated with the non perturbed circular vortex ring. The central line is then $\boldsymbol{X} = \boldsymbol{X}_b + \boldsymbol{X}'$, where $\boldsymbol{X}_b$ is a circle and $\boldsymbol{X}'$ its perturbation. It is of the form $\boldsymbol{X}' = \rho(t,\theta)\boldsymbol{i}_r(\theta) + \xi(t,\theta)\boldsymbol{e}_3$, where ρ and ξ are its *amplitudes* and $(\boldsymbol{i}_r, \boldsymbol{i}_\theta, \boldsymbol{e}_3)$ are the vectors of the cylindrical coordinates. WS [2] have studied the behaviour of this perturbation in the linear limit of small amplitudes. They used an equation of motion that comes from an ad-hoc regularisation of the singular Biot and Savart integral on $\mathcal{C}$ by a cut-off method. To extend their work, we study here these perturbations using the CT equation (1), which is linked to the inner structure of the filament contrary to the cut-off equation. The link to the inner structure is shown with a possible extension to a viscous vortex with an axial velocity. Here, we also give an analytical expression of the period of the linear oscillations. The non-linear behaviour of these perturbations are then studied numerically with (1) and a comparison between the numerics and the analytical study is given in the small amplitude limit. Finally, the simulation of a finite amplitude is performed to show the departure that introduces the non-linearity to the linear results.

2. Analytical study of the perturbation in the small amplitude limit

In the limit of small amplitude the equations become linear and have a solution of the form $\rho(t,\theta) = \rho(t)\,\mathrm{e}^{\mathrm{i}n\theta}$, $\xi(t,\theta) = \xi(t)\,\mathrm{e}^{\mathrm{i}n\theta}$, where $\mathrm{i}^2 = -1$ and n is an integer. These amplitudes follow the linear equations

$$\dot{\rho} = V_\xi \xi, \quad \dot{\xi} = V_\rho \rho \tag{2}$$

where $V_\xi = [-n^2\widetilde{V}_\alpha + g_\xi(n)]/(4\pi)$ and $V_\rho = [(n^2-1)\widetilde{V}_\alpha + g_\rho(n)]/(4\pi)$. Here the inner structure of the filament is given by the parameter $\widetilde{V}_\alpha$ which is 4π times the velocity of the circular vortex ring: in this linear regime, the leading order inner structure has the same evolution as the non-perturbed vortex ring.

Table. Values of $g_\xi(n)$ and $g_\rho(n)$ for modes between 2 and 6.

Tableau. *Valeurs de* $g_\xi(n)$ *et* $g_\rho(n)$ *pour des modes entre* 2 *et* 6.

n	2	3	4	5	6
$g_\xi(n)$	6.66	20.80	43.12	74.18	114.41
$g_\rho(n)$	−7.00	−21.33	−43.80	−74.97	−115.29

There is no vortex stretching; the axial velocity acts through C_w and the viscosity through the coefficient of diffusion α in the thickness $\bar{\delta}(t) = \sqrt{1+4\alpha^2 t}$ that is in C_v and C_w. The values of $g_\xi(n)$ and $g_\rho(n)$ are reported in the *table*.

For the inviscid vortex ($\alpha = 0$) $\widetilde{V}_0$ is a constant and the solution is of the form $\rho(t) = \rho_0\,\mathrm{e}^{\mathrm{i}\omega t}$ and $\xi(t) = \xi_0\,\mathrm{e}^{\mathrm{i}\omega t}$. One can easily deduce that $\omega = \pm\sqrt{-V_\rho V_\xi}$. The period $T = 2\pi/\omega$ of the oscillations is therefore:

$$T = 8\pi^2 \Big/ \sqrt{\left[n^2\widetilde{V}_0 - g_\xi(n)\right]\left[(n^2-1)\widetilde{V}_0 + g_\rho(n)\right]} \tag{3}$$

where $g_\xi(n)$ and $g_\rho(n)$ are given in the *table*, with $\widetilde{V}_0 = \ln(8/\varepsilon) - 1 + C_v(0) - 2m_0^2$. For $n = 2$ the expression of the period is in good agreement with the result of Dhanak and De-Bernardinis [4] based on WS results. The formula (3) generalizes that of Ricca [5] because it takes into account the non-local contribution (terms g_ξ and g_ρ) and an axial velocity. The evolution of the mode which is initially in a plane is:

$$\boldsymbol{X}'(\theta,t) = 2\rho_0\cos(n\theta)\left[\cos(\omega t)\boldsymbol{i}_r(\theta) + \frac{V_\rho}{\omega}\sin(\omega t)\boldsymbol{e}_3\right] \tag{4}$$

In the viscous case $\widetilde{V}_\alpha$ depends on the time and thus (2) has to be solved numerically. A fourth-order Runge–Kutta method has been used. The equation for ρ is $\ddot{\rho} - \dot{\rho}\dot{V}_\xi/V_\xi = V_\xi(t)V_\rho(t)\rho$. Thus the viscosity adds a damping term in the equation of ρ and now $-V_\xi(t)V_\rho(t)$ decreases with time.

3. Numerical study and comparison with the analytical study

A numerical code has been developed to simulate the full nonlinear equation of motion (1) of a *similar* vortex ring. An implicit finite difference scheme is used and the Simpson's rule is employed to integrate $\boldsymbol{Q}$. The temporal evolution of a vortex of initial shape in a plane $\boldsymbol{X}_0(\theta) = (1 + 2\rho_0\cos(n\theta))\boldsymbol{i}_r$ can be obtained for an inviscid or a viscous vortex. The numerical simulations of the viscous ($\alpha = 1$) mode $n = 2$ give the same result as Liu, Tavantziz and Ting [6].

Figure 1 shows the temporal evolution of circular vortex perturbed by the third inviscid mode for $\varepsilon = 0.02$, $\alpha = 0$, $\rho_0 = 0.05$ and $m_0 = 0$. This evolution is in good agreement with the linear eigenmode

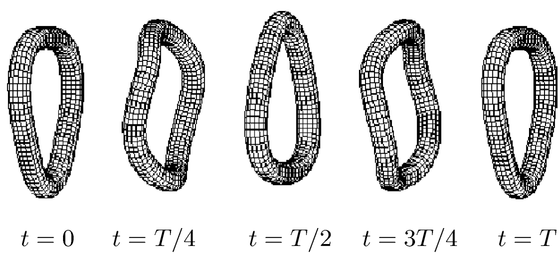

Figure 1. Temporal evolution (from left to right) of the inviscid mode $n = 3$ of small amplitude.

Figure 1. *Évolution temporelle (de gauche à droite) du mode inviscide* $n = 3$ *de faible amplitude.*

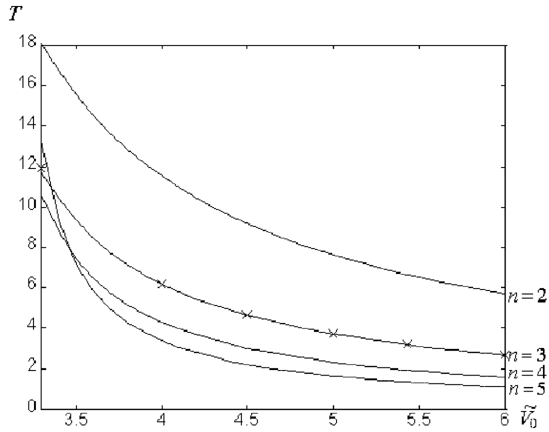

Figure 2. The period T of oscillations as a function of $\widetilde{V}_0$ and n obtained by the linear study. The crosses are the results of numerical simulations ($\rho_0 = 0.05$) for $n = 3$.

Figure 2. *Période d'oscillation en fonction de $\widetilde{V}_0$ et du nombre de mode n obtenue par l'étude linéaire. Les croix sont les résultats de simulations numériques* ($\rho_0 = 0{,}05$) *pour* $n = 3$.

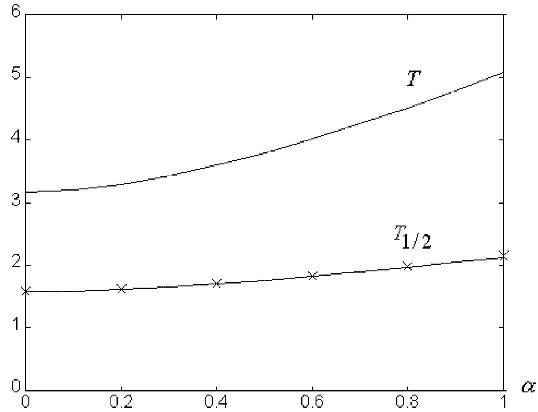

Figure 3. Pseudo-period T and half pseudo-period $T_{1/2}$, obtained by the linear study, as a function of the viscous parameter α for mode $n = 3$ and $\varepsilon = 0.02$. The crosses are the results of numerical simulations ($\rho_0 = 0.05$) for $n = 3$.

Figure 3. *Pseudo-période T et demie pseudo-période $T_{1/2}$, obtenues par l'étude linéaire, en fonction du paramètre de viscosité α pour $n = 3$ et $\varepsilon = 0{,}02$. Les croix sont les résultats de simulations numériques* ($\rho_0 = 0{,}05$) *pour* $n = 3$.

solution (4). It shows that intersection points between the circular and the perturbed central line stay always on the circle and that the points at fixed θ rotate around the circular vortex. *Figure 2* gives the period of inviscid oscillations as a function of $\widetilde{V}_0$ and n obtained by (3). A comparison with a numerical simulation ($\rho_0 = 0.005$) for the third mode is made. A viscous vortex, initially in a plane, is again in a plane at time $T_{1/2}$, called the *half-pseudo-period*, and at time T, called the *pseudo-period*. *Figure 3* gives the pseudo-period of the vortex ($\varepsilon = 0.02$) as a function of the viscous parameter α for a third mode. It is obtained from a numerical resolution of the linear equations with a Runge–Kutta solver. A comparison with a numerical simulation ($\rho_0 = 0.005$) for the third mode is performed and again, numerical and linear studies results match very well. In *figure 4* the temporal evolution of circular vortex perturbed by of a third mode of finite amplitude $\rho_0 = 0.125$ is performed with $\varepsilon = 0.02$, $\alpha = 0$ and $m_0 = 0$ in order to see the influence of the non-linear terms. We found that they decrease the velocity of the vortex from $5.43/(4\pi)$ to $4.97/(4\pi)$.

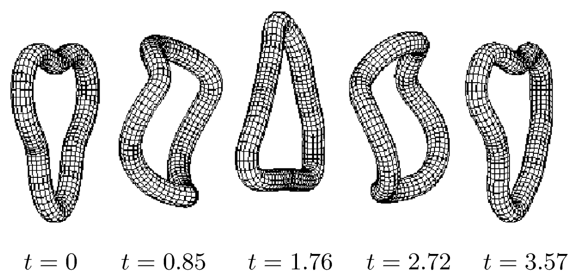

Figure 4. Temporal evolution (from left to right) of the inviscid mode $n = 3$ of finite amplitude.

Figure 4. *Évolution temporelle (de gauche à droite) du mode inviscide $n = 3$ d'amplitude finie.*

4. Conclusion

In this paper, the equation of motion of Callegari and Ting, that was derived from the Navier–Stokes equations, is used to study the long bending distortions of a slender circular vortex ring. The link between the evolution of the perturbation and the inner structure is given for an invicid or a viscous vortex. For an inviscid vortex, an analytical expression of the period of small oscillations is obtained and for a viscous vortex this period is found in solving the linear equations with a Runge–Kutta solver. A numerical code simulating the fully non-linear equation of motion has been developed. The simulation of small perturbations compares favorably with the linear results and finally the simulation of a finite perturbation shows the departure from the linear results induced by the non-linearities.

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