Champ de vitesse d'un filament tourbillon proche de la fibre

Le champ de vitesse créé par un filament tourbillon en un point $\underset{\sim}{x}$ de l'espace est donné par la formule :

$$\vec{v}(\underset{\sim}{x}) = \frac{\Gamma}{4\pi}\int_0^S -\frac{\vec{t}(s') \wedge (X(s') - \underset{\sim}{x})}{|X(s') - \underset{\sim}{x}|^3}\, ds'$$

Lorsqu'on est assez proche de la fibre pour utiliser les coordonnées locales à la fibre : $\underset{\sim}{x} = X(s) + r\vec{\pi}(\phi, s)$, l'expression précédente devient :

$$\vec{v}(r,\phi,s) = \frac{\Gamma}{4\pi}\int_0^S -\frac{\vec{t}(s') \wedge (X(s') - X(s) - r\vec{\pi}(\phi,s))}{|X(s') - X(s) - r\vec{\pi}(\phi,s)|^3}\, ds'$$

$$= \frac{\Gamma}{4\pi}\int_{-S^-}^{S^+} -\frac{\vec{t}(\bar{s}+s) \wedge (X(\bar{s}+s) - X(s) - r\vec{\pi}(\phi,s))}{|X(\bar{s}+s) - X(s) - r\vec{\pi}(\phi,s)|^3}\, d\bar{s}$$

$$= \frac{\Gamma}{4\pi}\int_{-S^-}^{S^+} K(\bar{s}, s, r, \phi)\, d\bar{s}$$

en posant $\bar{s} = s' - s$ et $S^+ = S^- = S/2$.

On voudrait obtenir un DA en fonction de r de ce champ de vitesse pour $r \simeq 0$.

Si on pose brutalement $r = 0$ dans cette expression on obtient une intégrale divergente au voisinage de $\bar{s} = 0$. Comme la perturbation est singulière, on va se servir de la DAR en faisant apparaître deux domaines distincts d'intégration (cf François).

On écrit donc $\vec{v}(r,\phi,s) = E_r^- + I_r^- + I_r^+ + E_r^+$ où :

$$E_r^- = \int_{-s^-}^{-\eta} K\, d\bar{s} \quad I_r^- = \int_{-\eta}^{0} K\, d\bar{s} \quad I_r^+ = \int_{0}^{\eta} K\, d\bar{s} \quad E_r^+ = \int_{\eta}^{s^+} K\, d\bar{s}$$

où l'on choisit de prendre $\boxed{1 \gg \eta \gg r}$

On pose $E_r = E_r^- + E_r^+$ et $I_r = I_r^- + I_r^+$

On fait le changement de variable $\tilde{s} = \bar{s}/r$ dans les intégrales intérieures I_r^- et I_r^+ :

$$I_r^+ = r \int_0^{\eta/r} - \frac{\vec{c}\,'(r\tilde{s}+s) \wedge (X(r\tilde{s}+s) - X(s) - r\vec{\pi}(\phi,s))}{|X(r\tilde{s}+s) - X(s) - r\vec{\pi}(\phi,s)|^3}\, d\tilde{s}$$

$$= r \int_0^{\eta/r} L(r,\tilde{s},s,\phi)\, d\tilde{s}$$

$$I_r^- = r \int_{-\eta/r}^{0} L(r,\tilde{s},s,\phi)\, d\tilde{s}.$$

I Le problème extérieur :

① Limite extérieure de $K(\bar{s},s,r,\phi)$:

c'est la limite : $\lim_{r\to 0}^{\bar{s}\text{ fixé}} K(\bar{s},s,r,\phi) = K_e$

- $[X(\bar{s}+s) - X(s) - r\vec{\pi}(\phi,s)] \cdot [X(\bar{s}+s) - X(s) - r\vec{\pi}(\phi,s)]$

$$= |X(\bar{s}+s) - X(s)|^2 - 2r\vec{\pi}\cdot(X(\bar{s}+s) - X(s)) + r^2$$

$$= \left\{ 1 - 2r\vec{\pi} \cdot \frac{(X(\bar{s}+s) - X(s))}{|X(\bar{s}+s) - X(s)|^2} + \frac{r^2}{|X(\bar{s}+s) - X(s)|^2} \right\} |X - X(s)|^2$$

D'où

$$|X(\bar{s}+s) - X(s) - r\vec{\pi}(\phi,s)|^{-3/2} = \left\{ 1 + 3r\vec{\pi} \cdot \frac{(X(\bar{s}+s) - X(s))}{|X(\bar{s}+s) - X(s)|^2} \right\} |X(\bar{s}+s) - X(s)|^{-3} + O(r^2)$$

$$\bullet \quad -\vec{\tau}(\bar{s}+s) \wedge \left\{ (X(\bar{s}+s) - X(s)) - r\,\vec{\pi}(\phi, s) \right\}$$

$$= -\vec{\tau}(\bar{s}+s) \wedge (X(\bar{s}+s) - X(s)) - r\,\vec{\pi}(\phi,s) \wedge \vec{\tau}(\bar{s}+s)$$

D'où

$$\boxed{K_e = \frac{-\vec{\tau}(\bar{s}+s) \wedge (X(\bar{s}+s) - X(s))}{|X(\bar{s}+s) - X(s)|^3} + \frac{r\,\vec{\pi}(\phi,s) \wedge -\vec{\tau}(\bar{s}+s)}{|X(\bar{s}+s) - X(s)|^3} - 3r\,\vec{\pi}(\phi,s) \cdot \frac{(X(\bar{s}+s) - X(s))}{|X(\bar{s}+s) - X(s)|^5}\, \vec{\tau}(\bar{s}+s) \wedge (X(\bar{s}+s) - X(s)) + O(r^2)}$$

$$= f(\bar{s}, s) + g(\bar{s}, s, r, \phi) + h(\bar{s}, s, r\,\phi) + O(r^2).$$

où $f(\bar{s},s) = -\dfrac{\vec{\tau}(\bar{s}+s) \wedge (X(\bar{s}+s) - X(s))}{|X(\bar{s}+s) - X(s)|^3}$. $\quad g(\bar{s},s,r,\phi) = \dfrac{r\,\vec{\pi} \wedge -\vec{\tau}(\bar{s}+s)}{|X(\bar{s}+s) - X(s)|^3}$

$$= r\,\vec{\pi}(s) \wedge g_1(\bar{s}, s)$$

$$h(\bar{s}, s, r, \phi) = -3r\,\vec{\pi}(\phi,s) \cdot \frac{(X(\bar{s}+s) - X(s))}{|X(\bar{s}+s) - X(s)|^5}\, \vec{\tau}(\bar{s}+s) \wedge (X(\bar{s}+s) - X(s)).$$

② **DA de E_r pour r proche de 0 :**

Posons : $F^+ = \displaystyle\int_{\eta}^{S^+} f(\bar{s},s)\,d\bar{s}$ $\quad F^- = \displaystyle\int_{-S^-}^{-\eta} f(\bar{s},s)\,d\bar{s}$ $\quad$ et $F = F^+ + F^-$

$G^+ = \displaystyle\int_{\eta}^{S^+} g(\bar{s},s,r,\phi)\,d\bar{s}$ $\quad G^- = \displaystyle\int_{-S^-}^{-\eta} g(\bar{s},s,r,\phi)\,d\bar{s}$ $\quad$ et $G = G^+ + G^-$

$H^+ = \displaystyle\int_{\eta}^{S^+} h(\bar{s},s,r,\phi)\,d\bar{s}$ $\quad H^- = \displaystyle\int_{-S^-}^{-\eta} h(\bar{s},s,r,\phi)\,d\bar{s}$ $\quad$ et $H = H^+ + H^-$

Comme $S^+ = S^-$, on laissera toute partie impaire de coté lors du calcul de F^+, G^+, H^+.

On cherche $E_{re} = \displaystyle\int_{-S^-}^{-\eta} K_e\,d\bar{s} + \int_{\eta}^{S^+} K_e\,d\bar{s} = F + G + H + O(r^2)$

2.1 Détermination de F :

2.1.1 Développement limité de f en $\bar{s} = 0$:

- $X(\bar{s}+s) - X(s) = X_s(s)\,\bar{s} + X_{ss}(s)\,\frac{\bar{s}^2}{2} + X_{sss}\,\frac{\bar{s}^3}{6} + o(\bar{s}^4)$

$$= \vec{\tau}(s)\,\bar{s} + \kappa(s)\,\vec{n}(s)\,\frac{\bar{s}^2}{2} + \left(\kappa_s\vec{n} + \kappa T\vec{b} - \kappa^2\vec{\tau}\right)\frac{\bar{s}^3}{6} + o(\bar{s}^4)$$

$$= \bar{s}\left\{\vec{\tau}(s) + \kappa\,\frac{\vec{n}}{2}\,\bar{s} + \left(\kappa_s\vec{n} + \kappa T\vec{b} - \kappa^2\vec{\tau}\right)\frac{\bar{s}^2}{6} + o(\bar{s}^3)\right\}$$

- $|X(\bar{s}+s) - X(s)|^2 = \left(\bar{s} - \frac{\kappa^2}{6}\bar{s}^3\right)^2 + \left(\kappa\,\frac{\bar{s}^2}{2} + \kappa_s\,\frac{\bar{s}^3}{6}\right) + \cancel{\left(\frac{\kappa T\bar{s}^3}{6}\right)^2} + o(\bar{s}^5)$

$$= \bar{s}^2 - \frac{\kappa^2}{3}\bar{s}^4 + \frac{\kappa^2}{4}\bar{s}^4 + o(\bar{s}^5)$$

$$= \bar{s}^2\left(1 - \bar{s}^2\,\frac{\kappa^2}{12} + o(\bar{s}^3)\right)$$

$$|X(\bar{s}+s) - X(s)|^{-3} = |\bar{s}|^{-3}\left(1 + \frac{3}{2}\,\frac{\bar{s}^2\kappa^2}{12} + o(\bar{s}^3)\right).$$

- $-\vec{\tau}(\bar{s}+s) = -\vec{\tau}(s) - \tau_s\,\bar{s} - \tau_{ss}\,\frac{\bar{s}^2}{2} + o(\bar{s}^3)$

$$= -\vec{\tau}(s) - \kappa(s)\,\vec{n}\,\bar{s} - \left(\kappa_s\vec{n} + \kappa T\vec{b} - \kappa^2\vec{\tau}\right)\frac{\bar{s}^2}{2} + o(\bar{s}^3)$$

- $-\tau\wedge\left(X(\bar{s}+s) - X(s)\right)$

$$= \bar{s}\left\{\kappa\,\vec{b}\,\bar{s} - \frac{\kappa}{2}\,\vec{b}\,\bar{s} - \vec{\tau}\wedge\left(\kappa_s\vec{n} + \kappa T\vec{b} - \kappa^2\vec{\tau}\right)\frac{\bar{s}^2}{6} - \left(\kappa_s\vec{n} + \kappa T\vec{b} - \kappa^2\vec{\tau}\right)\wedge\tau(s)\,\frac{\bar{s}^2}{2} + o(\bar{s}^3)\right.$$

$$= \bar{s}^2\left\{\frac{\kappa}{2}\,\vec{b} + \frac{\bar{s}}{6}\left[\overbrace{-\kappa_s\vec{b} + \kappa T\vec{n} + 3\vec{b}\kappa_s - 3\kappa T\vec{n}}^{2(\kappa_s\vec{b} - \kappa T\vec{n})}\right] + o(\bar{s}^2)\right.$$

$$= \bar{s}^2\left\{\frac{\kappa}{2}\,\vec{b} + \frac{\bar{s}}{3}\left[\kappa_s\vec{b} - \kappa T\vec{n}\right] + o(\bar{s}^2)\right\}$$

d'où : $$-\frac{\tau(\sigma+s)\wedge\left(X(\sigma+s)-X(s)\right)}{|X(\sigma+s)-X(s)|^3} = \underbrace{\frac{\sigma^2}{|\sigma|^3}}_{|\sigma|^{-1}}\left\{\frac{\kappa}{2}\vec{b} + \frac{\sigma}{3}\left[\kappa_s\vec{b}-\kappa T\vec{n}\right]+O(\sigma^2)\right\}$$

$$\boxed{-\frac{\tau(\sigma+s)\wedge\left(X(\sigma+s)-X(s)\right)}{|X(\sigma+s)-X(s)|^3} = \frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}+O(1)}$$

Q12 Détermination de F :

$$F^+ = \int_\eta^{S^+} f(\sigma,s)\,d\sigma = \int_\eta^{S^+}\left[f(\sigma,s)-\frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}\right]d\sigma + \int_\eta^{S^+}\frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}\,d\sigma$$

fonction paire

$$F = \int_{-S^-}^{-\eta}\left[f(\sigma,s)-\frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}\right]d\sigma + \int_\eta^{S^+}\left[f(\sigma,s)-\frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}\right]d\sigma + 2\int_\eta^{S^+}\frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}\,ds$$

$$2\int_\eta^{S^+}\frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}\,d\sigma = \kappa\left[\ln S^+ - \ln\eta\right]\vec{b}$$

$$\int_\eta^{S^+}\left(f(\sigma,s)-\frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}\right]d\sigma = \int_0^{S^+}\left[f(\sigma,s)-\frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}\right]d\sigma + \eta \underset{\text{impair}}{\alpha} + O(\eta^2)$$

$$F = \int_{-S^-}^{S^+}\left[f(\sigma,s)-\frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}\right]d\sigma - \kappa(s)\vec{b}(s)\ln\frac{\eta}{S^+} + O(\eta^2)$$

Soit $\boxed{\vec{A} = \int_{-S^-}^{S^+}\left[f(\sigma,s)-\frac{\kappa}{2}\frac{\vec{b}}{|\sigma|}\right]d\sigma}$

on a $\boxed{F = \vec{A} - \kappa(s)\vec{b}(s)\ln\frac{\eta}{S^+} + O(\eta^2)}$

$\boxed{\ln\eta \gg 1 \gg \eta \gg r \gg r\eta}$

$r \gg \eta^2$ si $\sqrt{r} \gg \eta$ → on prend η tel que $\boxed{\sqrt{r} \gg \eta \gg r}$

2.2 Détermination de G :

$$G^+ = r\vec{\pi}(\phi,s)\wedge\int_{\eta}^{s^+} g_1(\bar{s},s)\,d\bar{s} \qquad G_1^+ = \int_{\eta}^{s^+} g_1(\bar{s},s)\,d\bar{s} \quad G_1^- = \int_{-s^-}^{-\eta} g_1(\bar{s},s)\,d\bar{s}.$$

$$G^- = r\vec{\pi}(\phi,s)\wedge\int_{-s^-}^{-\eta} g_1(\bar{s},s)\,d\bar{s} \qquad G = G^- + G^+ = r\vec{\pi}(\phi,s)\wedge G_1$$

2.2.1 Développement limité de g_1 en $\bar{s}=0$:

On a (cf 2.1.1) :

$$-\vec{\tau}(\bar{s}+s) = -\vec{\tau}(s) - K(s)\vec{n}\,\bar{s} - (K_s\vec{n} + KT\vec{b} - K^2\vec{\tau})\frac{\bar{s}^2}{2} + O(\bar{s}^3$$

$$|X(\bar{s}+s) - X(s)|^{-3} = |\bar{s}|^{-3}\left(1 + \frac{3}{2}\,\frac{\bar{s}^2 K^2}{12} + O(\bar{s}^3)\right)$$

D'où :

$$\frac{-\vec{\tau}(\bar{s}+s)}{|X(\bar{s}+s) - X(s)|^3} = \frac{1}{|\bar{s}|^3}\Big[-\vec{\tau}(s) - K(s)\vec{n}\,\bar{s} - (K_s\vec{n} + KT\vec{b} - K^2\vec{\tau})\frac{\bar{s}^2}{2} - \frac{3}{2}\frac{K^2}{12}\vec{\tau}(s)\bar{s}^2 + O(\bar{s}^3)\Big]$$

2.2.2 Determination de G :

$$G_1^+ = \int_{\eta}^{s^+}\left\{-\frac{\vec{\tau}(s+\bar{s})}{|X(s+\bar{s}) - X(s)|^3} + \frac{1}{|\bar{s}|^3}\Big[\vec{\tau}(s) + K(s)\vec{n}\,\bar{s} + (K_s\vec{n} + KT\vec{b} - \frac{3}{4}K^2\vec{\tau})\frac{\bar{s}^2}{2}\Big]\right\}d\bar{s}$$

$$-\vec{\tau}(s)\int_{\eta}^{s^+}\Big[\frac{1}{|\bar{s}|^3} - \frac{3}{8}K^2\frac{1}{|\bar{s}|}\Big]d\bar{s} - \vec{n}\int_{\eta}^{s^+}\Big[\overset{\text{impair}}{\downarrow}\,K(s)\frac{\bar{s}}{|\bar{s}|^3} + \frac{K_s}{2}\frac{1}{|\bar{s}|}\Big]d\bar{s} - \vec{b}\int_{\eta}^{s^+}\frac{KT}{2}\frac{1}{|\bar{s}|}d\bar{s}$$

$$= \int_0^{s^+}\left\{\frac{-\vec{\tau}(s+\bar{s})}{|X(s+\bar{s}) - X(s)|^3} + \frac{1}{|\bar{s}|^3}\Big[\vec{\tau}(s) + K(s)\vec{n}\,\bar{s} + (K_s\vec{n} + KT\vec{b} - \frac{3}{4}K^2\vec{\tau})\frac{\bar{s}^2}{2}\Big]\right\}d\bar{s} + O(\eta)$$

$$-\vec{\tau}(s)\left\{-\frac{1}{2}\Big[\frac{1}{\bar{s}^2}\Big]_{\eta}^{s^+} - \frac{3}{8}K^2\big[\ln\bar{s}\big]_{\eta}^{s^+}\right\} - \vec{n}\frac{K_s}{2}\big[\ln\bar{s}\big]_{\eta}^{s^+} - \vec{b}\frac{KT}{2}\big[\ln\bar{s}\big]_{\eta}^{s^+}$$

$$G_1 = \int_{-S^-}^{S^+} \left\{ \frac{-\vec{P}(s+\sigma)}{|X(s+\sigma)-X(s)|^3} + \frac{1}{|\sigma|^3}\left[\vec{P}(s) + K(s)\vec{n}\sigma + (K_s\vec{n} + KT\vec{b} - \frac{3}{4}K^2\vec{P})\frac{\sigma^2}{2}\right]\right\} d\sigma$$

$$- \vec{P}(s)\left\{\frac{1}{\eta^2} - \frac{1}{S^{+2}} + \frac{3}{4}K^2 \ln\frac{\eta}{S^+}\right\} + \vec{n}K_s \ln\frac{\eta}{S^+} + \vec{b}KT\ln\frac{\eta}{S^+} + O(\eta)$$

On a $\vec{P} \wedge \vec{\pi} = \vec{\Phi}$ $\quad \vec{n} \wedge \vec{\pi} = \sin\phi\, \vec{P}$ $\quad \vec{b} \wedge \vec{\pi} = -\vec{P}\cos\theta$

puisque $\vec{\pi} = \vec{n}\cos\phi + \vec{b}\sin\phi$.

Soit
$$\boxed{\vec{B} = \vec{\pi}(s) \wedge \int_{-S^-}^{S^+} \left\{ \frac{-\vec{P}(\sigma+s)}{|X(\sigma+s)-X(s)|^3} + \frac{1}{|\sigma|^3}\left[\vec{P}(s) + K(s)\vec{n}\sigma + (K_s\vec{n} + KT\vec{b} - \frac{3}{4}K^2\vec{P})\frac{\sigma^2}{2}\right]\right\} d\sigma}$$

$$G = r\vec{B} + r\vec{\Phi}\left\{\frac{1}{\eta^2} - \frac{1}{S^{+2}} + \frac{3}{4}K^2\ln\frac{\eta}{S^+}\right\} - r\sin\phi\,\vec{P}K_s\ln\frac{\eta}{S^+} + r\cos\phi\,\vec{P}KT\ln\frac{\eta}{S^+} + O(r\eta)$$

$$\boxed{G = r\vec{B} + r\vec{\Phi}\left\{\frac{1}{\eta^2} - \frac{1}{S^{+2}} + \frac{3}{4}K^2\ln\frac{\eta}{S^+}\right\} + r\vec{P}\left\{-\sin\phi K_s + \cos\phi KT\right\}\ln\frac{\eta}{S^+} + O(r\eta)}$$

2.3 Détermination de H :

2.3.1 Développement limité de h en $\sigma = 0$:

On a (ch 2.1.1) :

$$-P(s+\sigma) \wedge (X(s+\sigma) - X(s)) = \sigma^2\left\{\frac{K}{2}\vec{b} + \frac{\sigma}{3}[K_s\vec{b} - KT\vec{n}] + O(\sigma^2)\right\}$$

$$X(s+\sigma) - X(s) = \sigma\left\{\vec{P}(s) + K\frac{\vec{n}}{2}\sigma + (K_s\vec{n} + KT\vec{b} - K^2\vec{P})\frac{\sigma^2}{6} + O(\sigma^3)\right\}$$

A partir du développement de $|X(s+\sigma) - X(s)|^2$ il vient :

$$|X(s+\sigma) - X(s)|^{-5} = |\sigma|^{-5}\left(1 + \frac{5}{2}\sigma^2\frac{K^2}{12} + O(\sigma^3)\right)$$

Alors :

$$\vec{r}'(s)\cdot(X(s+\bar{s})-X(s)) = \bar{s}\left\{\frac{K}{2}\cos\varphi\,\bar{s} + (K_s\cos\phi + KT\sin\varphi)\frac{\bar{s}^2}{6} + O(\bar{s}^3)\right\}$$

$$= \bar{s}^2\left\{\frac{K}{2}\cos\varphi + (K_s\cos\varphi + KT\sin\varphi)\frac{\bar{s}}{6} + O(\bar{s}^2)\right\}$$

$\vec{r}'\cdot\vec{t}' = 0$ $\quad \vec{r}'\cdot\vec{n} = \cos\varphi$ $\quad \vec{r}'\cdot\vec{b}' = \sin\varphi$ puisque $\vec{r}' = \vec{n}\cos\varphi + \vec{b}'\sin\phi$.

$$\frac{\bar{s}^2\cdot\bar{s}^2}{|\bar{s}|^5} = \frac{1}{|\bar{s}|}$$

D'où :

$$-\vec{r}'(s)\cdot\frac{(X(s+\bar{s})-X(s))}{|X(s+\bar{s})-X(s)|^5}\left[\vec{t}'(\bar{s}+s)\wedge(X(s+\bar{s})-X(s))\right] = \frac{1}{|\bar{s}|}\left\{\frac{K}{2}\vec{b}'\frac{K}{2}\cos\varphi\right\} + O\left(\frac{\bar{s}}{|\bar{s}|}\right)$$

(2.3.2) Détermination de H :

$$H^+ = 3r\int_{\eta}^{S^+} -\vec{r}'(s)\cdot\frac{(X(s+\bar{s})-X(s))}{|X(s+\bar{s})-X(s)|^5}\left[\vec{t}'(\bar{s}+s)\wedge(X(\bar{s}+s)-X(s))\right]d\bar{s}$$

$$= 3r\int_{\eta}^{S^+}\left\{-\frac{r(s)\cdot(X(s+\bar{s})-X(s))}{|X(s+\bar{s})-X(s)|^5}\left[\vec{t}'(\bar{s}+s)\wedge(X(\bar{s}+s)-X(s))\right] - \left(\frac{K}{2}\right)^2\frac{\vec{b}}{|\bar{s}|}\cos\varphi\right\}d\bar{s}$$

$$-3r\left(\frac{K}{2}\right)^2\vec{b}'\cos\phi\left[\ln\frac{\eta}{S^+}\right]$$

Soit
$$\boxed{\vec{C} = \int_{-S^-}^{S^+}\left\{\vec{r}'(s)\cdot\frac{(X(s+\bar{s})-X(s))}{|X(s+\bar{s})-X(s)|^5}\left[\vec{t}'(\bar{s}+s)\wedge(X(s+\bar{s})-X(s))\right] + \left(\frac{K}{2}\right)^2\frac{\vec{b}\cos\varphi}{|\bar{s}|}\right\}d\bar{s}}$$

On a :

$$\boxed{H = -3r\vec{C} - 6r\left(\frac{K}{2}\right)^2\vec{b}\cos\varphi\left[\ln\frac{\eta}{S^+}\right] + O(r\eta)}$$

(2.4) Expression de E_{rl}, DA extérieur de E_r en $r=0$:

$$E_{rl} = F + G + H + O(r^2)$$

II Le problème intérieur:

(1) Limite intérieure de $L(\tilde s, s, r, \phi)$:

C'est la limite : $\lim\limits_{r\to 0}^{\tilde s \text{ fixé}} L(\tilde s, s, r, \phi) = L_e$

• $-\vec\tau(r\tilde s+s) = -\vec\tau(s) - \kappa(s)\vec n\, r\tilde s - (\kappa_s\vec n + \kappa T\vec b - \kappa^2\vec\tau)\frac{r^2\tilde s^2}{2} + O(r^3)$

en se servant du développement de $\vec\tau(\tilde s+s)$ en $\tilde s=0$ (cf 211).

• $X(r\tilde s+s) - X(s) = r\tilde s\left\{\vec\tau(s) + \frac{\kappa}{2}\vec n\, r\tilde s + (\kappa_s\vec n + \kappa T\vec b - \kappa^2\vec\tau)\frac{\tilde s^2 r^2}{6} + O(r^3)\right\}$ (cf 211)

$$X(r\tilde s+s) - X(s) - r\vec r(\phi,s) = r\left\{\tilde s\vec\tau - \vec r + \frac{\kappa}{2}\vec n\,\tilde s^2 r + (\kappa_s\vec n + \kappa T\vec b - \kappa^2\vec\tau)\frac{\tilde s^3}{6}r^2 + O(r^3)\right\}$$

• $\vec\tau\wedge\vec r = \vec\phi$ $\quad\vec n\wedge\vec r = \sin\phi\,\vec\tau$ $\quad\vec b\wedge\vec r = -\cos\phi\,\vec\tau$

puisque $\vec r = \cos\phi\,\vec n + \sin\phi\,\vec b$ et $\vec\phi = -\vec n\sin\phi + \vec b\cos\phi$

• $-\vec\tau(r\tilde s+s)\wedge\left(X(r\tilde s+s) - X(s) - r\vec r(\phi,s)\right)$

$$= -r\Big\{-\vec\tau\wedge\vec r + \kappa\tilde s^2\vec n\wedge\vec\tau\, r - \kappa\tilde s\,\vec n\wedge\vec r\, r + \frac{\kappa}{2}\tilde s^2\,\vec\tau\wedge\vec n\, r$$
$$+\vec\tau\wedge(\kappa_s\vec n + \kappa T\vec b)\frac{\tilde s^3}{6}r^2 + (\kappa_s\vec n - \kappa T\vec b - \kappa^2\vec\tau)\frac{\tilde s^2}{2}\wedge(\tilde s\vec\tau - \vec r)r^2 + O(r^3)\Big\}$$

$$= -r\Big\{-\vec\phi - \frac{\kappa}{2}\tilde s^2\vec b\, r - \kappa\sin\phi\,\tilde s\,\vec\tau\, r + \kappa_s\frac{\tilde s^3}{6}\vec b\, r^2 - \frac{\kappa T}{6}\tilde s^3\vec n\, r^2 + \kappa^2\frac{\tilde s^2}{2}r^2\vec\phi$$
$$-\kappa_s\frac{\tilde s^3}{2}\vec b\, r^2 + \frac{\kappa T}{2}\tilde s^3\vec n\, r^2 - \kappa_s\frac{\tilde s^2}{2}\sin\phi\,\vec\tau\, r^2 + \kappa T\frac{\tilde s^2}{2}\cos\phi\,\vec\tau\, r^2 + O(r^3)\Big\}$$

$$= -r\Big\{-\vec\phi - \left(\frac{\kappa}{2}\tilde s^2\vec b + \kappa\sin\phi\,\tilde s\,\vec\tau\right)r + \Big(-\frac{\kappa_s\tilde s^3}{3}\vec b + \frac{1}{3}\kappa T\tilde s^3\vec n + \kappa^2\frac{\tilde s^2}{2}r^2\vec\phi$$
$$+\frac{\tilde s^2}{2}\left[\kappa T\cos\phi - \kappa_s\sin\phi\right]\vec\tau\Big)r^2 + O(r^3)\Big\}$$

$\bullet\ \left(X(r\tilde{s}+s)-X(s)-r\vec{\pi}(\phi,s)\right)\cdot\left(X(r\tilde{s}+s)-X(s)-r\vec{\pi}(\phi,s)\right)$

$$= r^2\left\{ |\tilde{s}\vec{t}-\vec{\pi}|^2 - \kappa\tilde{s}^2\vec{\pi}\cdot\vec{n}\, r + \frac{\tilde{s}^3}{3}[\tilde{s}\vec{t}-\vec{\pi}]\cdot[\kappa_s\vec{n}+\kappa T\vec{b}-\kappa^2\vec{t}]r^2 + \frac{\kappa^2}{4}\tilde{s}^4 r^2 + O(r^3)\right\}$$

$$= r^2\left\{1+\tilde{s}^2-\kappa\tilde{s}^2\cos\phi\, r + \frac{\tilde{s}^3}{3}\left[-\kappa_s\cos\phi-\kappa T\sin\phi-\tilde{s}\frac{\kappa^2}{4}\right]r^2 + O(r^3)\right\}$$

$$\left|X(r\tilde{s}+s)-X(s)-r\vec{\pi}(\phi,s)\right|^{-3}$$

$$= \frac{1}{r^3}\frac{1}{(1+\tilde{s}^2)^{3/2}}\left\{1+\frac{3}{2}\frac{\kappa\tilde{s}^2}{1+\tilde{s}^2}\cos\phi\, r - \frac{\tilde{s}^3}{2}\frac{\left[-\kappa_s\cos\phi-\kappa T\sin\phi-\tilde{s}\frac{\kappa^2}{4}\right]}{1+\tilde{s}^2}r^2 + \frac{15}{8}\frac{\kappa^2\tilde{s}^4\cos^2\phi}{(1+\tilde{s}^2)^2}r^2 + O(r^3)\right\}$$

car $(1+\varepsilon)^{-3/2} = 1-\frac{3}{2}\varepsilon+\frac{15}{4}\frac{\varepsilon^2}{2}$

$\bullet$ On a donc :

$$\boxed{\begin{aligned} r\, L_e = -\frac{1}{r}\frac{1}{(1+\tilde{s}^2)^{3/2}}\Bigg\{ & -\vec{\phi}-\frac{3}{2}\frac{\kappa\tilde{s}^2}{1+\tilde{s}^2}\cos\phi\,\vec{\phi}\, r - \left(\frac{\kappa}{2}\tilde{s}^2\vec{b}+\kappa\sin\phi\,\tilde{s}\vec{t}\right)r \\ & + \frac{\tilde{s}^3}{2}\frac{-\kappa_s\cos\phi-\kappa T\sin\phi-\tilde{s}\frac{\kappa^2}{4}}{1+\tilde{s}^2}\vec{\phi}\, r^2 - \frac{15}{8}\kappa^2\tilde{s}^4\frac{\cos^2\phi}{(1+\tilde{s}^2)^2}\vec{\phi}\, r^2 \\ & - \left(\frac{\kappa}{2}\tilde{s}^2\vec{b}+\kappa\sin\phi\,\tilde{s}\vec{t}\right)\frac{3}{2}\frac{\kappa\tilde{s}^2\cos\phi}{1+\tilde{s}^2}r^2 \\ & + \left(-\frac{\kappa_s\tilde{s}^3}{3}\vec{b}+\frac{1}{3}\kappa T\tilde{s}^3\vec{n}+\frac{\tilde{s}^2}{2}[\kappa T\cos\phi-\kappa_s\sin\phi]\vec{t}+\kappa^2\frac{\tilde{s}^2}{2}\vec{\phi}\right)r^2 + O(r^3)\Bigg\} \end{aligned}}$$

$$= f' + g' + h' + O\left(\frac{r^2}{(1+\tilde{s}^2)^{3/2}}\right)$$

où $f' = \dfrac{\vec{\phi}}{r(1+\tilde{s}^2)^{3/2}}$ $\quad g' = \dfrac{3}{2}\dfrac{\kappa\tilde{s}^2}{(1+\tilde{s}^2)^{5/2}}\cos\phi\,\vec{\phi} + \dfrac{\frac{\kappa}{2}\tilde{s}^2\vec{b}+\overset{\text{impair}}{\overset{\downarrow}{\kappa\sin\phi\,\tilde{s}\vec{t}}}}{(1+\tilde{s}^2)^{3/2}}$

$$h' = r\left\{\frac{\tilde{s}^4\frac{\kappa^2}{4}\vec{\phi}}{(1+\tilde{s}^2)^{5/2}} + \frac{15}{8}\kappa^2\tilde{s}^4\frac{\cos^2\phi\ \vec{\phi}}{(1+\tilde{s}^2)^{7/2}} + \left(\frac{\kappa}{2}\tilde{s}^2\vec{b}\right)\frac{3}{2}\frac{\kappa\tilde{s}^2\cos\phi}{(1+\tilde{s}^2)^{5/2}} - \frac{\frac{\tilde{s}^2}{2}[\kappa T\cos\phi-\kappa_s\sin\phi]\vec{t}+\frac{\tilde{s}^2}{2}\kappa^2\vec{\phi}}{(1+\tilde{s}^2)^{3/2}}\right\}$$

en laissant les termes impairs.

② DA de I_r pour r proche de 0 :

Posons

$$F'^+ = \int_0^{\eta/r} f' d\tilde{s} \qquad F'^- = \int_{-\eta/r}^0 f' d\tilde{s} \qquad \text{et } F' = F'^- + F'^+$$

$$G'^+ = \int_0^{\eta/r} g' d\tilde{s} \qquad G'^- = \int_{-\eta/r}^0 g' d\tilde{s} \qquad \text{et } G' = G'^- + G'^+$$

$$H'^+ = \int_0^{\eta/r} h' d\tilde{s} \qquad H'^- = \int_{-\eta/r}^0 \cancel{f}' d\tilde{s} \qquad \text{et } H' = H'^- + H'^+$$

On cherche $\displaystyle I_{re} = \int_{-\eta/r}^0 r L e d\tilde{s} + \int_0^{\eta/r} r L e d\tilde{s}$

$$= F' + G' + H' + O(r^2)$$

②1 Détermination de F' :

$$F' = \int_{-\eta/r}^0 f' d\tilde{s} + \int_0^{\eta/r} f' d\tilde{s} = \frac{2}{r} \vec{\Phi}(1) \int_0^{\eta/r} \frac{d\tilde{s}}{(1+\tilde{s}^2)^{3/2}} = \frac{2}{r} \vec{\Phi}(1) \left[\frac{\tilde{s}}{(1+\tilde{s}^2)^{1/2}} \right]_0^{\eta/r}$$

$$= 2 \frac{\eta}{r^2} \vec{\Phi}(1) \left(1 + \left(\frac{\eta}{r}\right)^2\right)^{-1/2} \qquad (1+\varepsilon)^{-1/2} = 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2$$

$$= 2 \frac{\vec{\Phi}(1)}{r} \left(1 - \frac{1}{2}\left(\frac{r}{\eta}\right)^2 + \frac{3}{4}\left(\frac{r}{\eta}\right)^4 + O\left(\left(\frac{r}{\eta}\right)^6\right)\right)$$

$$\boxed{F' = \frac{2\vec{\Phi}(1)}{r} - \vec{\Phi}(1)\frac{r}{\eta^2} + \frac{3}{4}\vec{\Phi}\frac{r^3}{\eta^4} + O\left(\frac{r^5}{\eta^6}\right)}$$

$r \gg \frac{r^5}{\eta^6} \Rightarrow \eta \gg r^{2/3}$ Or $r^{1/2} \gg r^{2/3} \gg r$ car $1/2 < \frac{2}{3} < 1$

On prend η dans l'intervalle suivant : $\boxed{r^{1/2} \gg \eta \gg r^{2/3}}$

(22) <u>Détermination de G'</u> :

$$G' = \int_{-n/r}^{0} g'\,d\tilde{s} + \int_{0}^{n/r} g'\,d\tilde{s} = 3K\cos\phi\,\vec{p}\int_0^{n/r}\frac{\tilde{s}^2}{(1+\tilde{s}^2)^{5/2}}\,d\tilde{s} + K\vec{b}\int_0^{n/r}\frac{\tilde{s}^2}{(1+\tilde{s}^2)^{3/2}}\,d\tilde{s}$$

Or $\displaystyle\int_0^{n/r}\frac{\tilde{s}^2}{(1+\tilde{s}^2)^{5/2}}\,d\tilde{s} = \frac{1}{3}\left[\frac{\tilde{s}^3}{(1+\tilde{s}^2)^{3/2}}\right]_0^{n/r} = \frac{1}{3}\,\frac{(n/r)^3}{\left[1+(n/r)^2\right]^{3/2}} = \frac{1}{3}\,\frac{1}{\left[1+\left(\frac{r}{n}\right)^2\right]^{3/2}}$

$$\left(1+\left(\frac{r}{n}\right)^2\right)^{-3/2} = 1-\frac{3}{2}\left(\frac{r}{n}\right)^2+\frac{15}{8}\left(\frac{r}{n}\right)^4+o\left(\left(\frac{r}{n}\right)^6\right)$$

$$\int_0^{n/r}\frac{\tilde{s}^2\,d\tilde{s}}{(1+\tilde{s}^2)^{5/2}} = \frac{1}{3}\left[1-\frac{3}{2}\left(\frac{r}{n}\right)^2+o\left(\frac{r}{n}\right)^4\right].$$

$r \gg \left(\frac{r}{n}\right)^4$ si $n^4 \gg r^3 \Leftrightarrow n \gg r^{3/4}$ or $r^{2/3} \gg r^{3/4}$ et $n \gg r^{2/3}$.

$$\int_0^{n/r}\frac{\tilde{s}^2\,d\tilde{s}}{(1+\tilde{s}^2)^{3/2}} = \left[-\frac{\tilde{s}}{(1+\tilde{s}^2)^{1/2}}+\ln\left(\tilde{s}+\sqrt{1+\tilde{s}^2}\right)\right]_0^{n/r}$$

$$= -\frac{n/r}{(1+(n/r)^2)^{1/2}}+\ln\left(\frac{n}{r}+\sqrt{1+(n/r)^2}\right)$$

$$= -\frac{1}{\left(1+\left(\frac{r}{n}\right)^2\right)^{1/2}}+\ln\frac{n}{r}+\ln\left(1+\sqrt{1+(r/n)^2}\right)$$

$\left(1+\left(\frac{r}{n}\right)^2\right)^{-\frac{1}{2}} = 1-\frac{1}{2}\left(\frac{r}{n}\right)^2+\frac{3}{8}\left(\frac{r}{n}\right)^4+o\left(\left(\frac{r}{n}\right)^6\right)$ $\qquad (1+\varepsilon)^{-1/2} = 1-\frac{\varepsilon}{2}$

$$\left(1+\left(\frac{r}{n}\right)^2\right)^{1/2} = 1+\frac{1}{2}\left(\frac{r}{n}\right)^2+o\left(\left(\frac{r}{n}\right)^4\right)$$

$$\ln\left(1+\sqrt{1+(r/n)^2}\right) = \ln\left(2+\frac{1}{2}\left(\frac{r}{n}\right)^2+o\left(\left(\frac{r}{n}\right)^4\right)\right) = \ln 2+\ln\left(1+\frac{1}{4}\left(\frac{r}{n}\right)^2+o\left(\frac{r}{n}\right)^4\right)$$

$$= \ln 2+\frac{1}{4}\left(\frac{r}{n}\right)^2+o\left(\frac{r}{n}\right)^4 \quad \text{car } \ln(1+\varepsilon) = \varepsilon+o(\varepsilon^2)$$

D'où $\displaystyle\int_0^{n/r}\frac{\tilde{s}^2}{(1+\tilde{s}^2)^{3/2}}\,d\tilde{s} = -1+\frac{1}{2}\left(\frac{r}{n}\right)^2+o\left(\left(\frac{r}{n}\right)^4\right)+\ln\left(\frac{n}{r}\right)+\ln 2+\frac{1}{4}\left(\frac{r}{n}\right)^2+o\left(\frac{r}{n}\right)^4$

$$= -1+\ln\frac{n}{r}+\ln 2+\frac{3}{4}\left(\frac{r}{n}\right)^2+o\left(\frac{r}{n}\right)^4$$

Il vient donc :

$$\boxed{G' = K\cos\phi\,\vec{\phi}\left[1-\frac{3}{2}\left(\frac{r}{\eta}\right)^2\right] + K\vec{b}\left[-1+\ln 2+\ln\frac{\eta}{r}+\frac{3}{4}\left(\frac{r}{\eta}\right)^2\right]+O\left(\left(\frac{r}{\eta}\right)^4\right)}$$

(23) Détermination de H' : $H' = \int_{-\eta/r}^{0} h'\,d\tilde{s} + \int_{0}^{\eta/r} h'\,d\tilde{s}$

$$H' = -2r\left\{\left[-\frac{K^2}{8}\vec{\phi}-\frac{3}{4}K^2\cos\phi\,\vec{b}\right]\int_0^{\eta/r}\frac{\tilde{s}^4}{(1+\tilde{s}^2)^{5/2}}d\tilde{s} + \frac{K^2}{2}\vec{\phi}\int_0^{\eta/r}\frac{\tilde{s}^2}{(1+\tilde{s}^2)^{3/2}}d\tilde{s}\right.$$

$$\left. -\frac{15}{8}K^2\cos^2\phi\,\vec{\phi}\int_0^{\eta/r}\frac{\tilde{s}^4}{(1+\tilde{s}^2)^{7/2}}d\tilde{s} + \left[KT\cos\phi - K_s\sin\phi\right]\frac{\vec{c}}{2}\int_0^{\eta/r}\frac{\tilde{s}^2}{(1+\tilde{s}^2)^{3/2}}d\tilde{s}\right\}$$

$$\int_0^{\eta/r}\frac{\tilde{s}^4}{(1+\tilde{s}^2)^{5/2}}d\tilde{s} = \left[-\frac{\tilde{s}}{\sqrt{1+\tilde{s}^2}}-\frac{1}{3}\frac{\tilde{s}^3}{(1+\tilde{s}^2)^{3/2}}+\ln\left(\tilde{s}+\sqrt{1+\tilde{s}^2}\right)\right]_0^{\eta/r}$$

$$= -1+O\left(\left(\frac{r}{\eta}\right)^2\right)-\frac{1}{3}+O\left(\left(\frac{r}{\eta}\right)^2\right)+\ln\frac{\eta}{r}+\ln 2+O\left(\left(\frac{r}{\eta}\right)^2\right)$$

$$= -\frac{4}{3}+\ln 2+\ln\frac{\eta}{r}+O\left(\frac{r}{\eta}\right)^2$$

On a $r \gg r\left(\frac{r}{\eta}\right)^2$

$$\int_0^{\eta/r}\frac{\tilde{s}^2}{(1+\tilde{s}^2)^{3/2}}d\tilde{s} = -1+\ln\eta/r+\ln 2+O\left(\left(\frac{r}{\eta}\right)^2\right)$$ d'après 22.

$$\int_0^{\eta/r}\frac{\tilde{s}^4}{(1+\tilde{s}^2)^{7/2}}d\tilde{s} = \left[\frac{1}{5}\frac{\tilde{s}^5}{(1+\tilde{s}^2)^{5/2}}\right]_0^{\eta/r}$$

$$= \frac{1}{5}\left(1+\left(\frac{r}{\eta}\right)^2\right)^{-5/2} = \frac{1}{5}\left(1+O\left(\left(\frac{r}{\eta}\right)^2\right)\right)$$

D'où

$$\boxed{\begin{aligned} H' = -2r\Big\{&\left[-\frac{K^2}{8}\vec{\phi}-\frac{3}{4}K^2\cos\phi\,\vec{b}\right]\left(-\frac{4}{3}+\ln 2+\ln\frac{\eta}{r}\right)-\frac{3}{8}K^2\cos^2\phi\,\vec{\phi} \\ &+\left[KT\cos\phi-K_s\sin\phi\right]\frac{\vec{c}}{2}\left(-1+\ln\frac{\eta}{r}+\ln 2\right)+\frac{K^2}{2}\vec{\phi}\left(-1+\ln\frac{\eta}{r}+\ln 2\right)\Big\}\end{aligned}}$$

(24) Expression de I_{rl}, DA ~~extérieur~~ intérieur de I_r en $r=0$:

$$I_{rl} = F'+G'+H'+O(r^2)$$

III. DA de $\vec{v}(r,\phi,s)$ en $r=0$:

On a $\vec{v}_\ell(r,\phi,s) = (E_{r\ell} + I_{r\ell})\frac{\Gamma}{4\pi}$

$$= (F+G+H+F'+G'+H'+O(r^2))\frac{\Gamma}{4\pi}$$

$$\frac{4\pi}{\Gamma}\vec{v}_\ell = \{\vec{A} - \boxed{K(s)\vec{b}(s)\ln\eta} + K(s)\vec{b}(s)\ln S^+ + O(\eta^2)\}$$

$$+\{r(\vec{B} - \frac{\vec{\phi}}{S^{+2}} - \frac{3}{4}K^2\ln S^+\vec{\phi} + \sin\phi\,\vec{C}K_s\ln S^+ - \cos\phi\,\vec{C}KT\ln S^+)$$

$$+\boxed{\vec{\phi}\frac{r}{\eta^2}} + \boxed{[\frac{3}{4}K^2\vec{\phi} - \sin\phi\,\vec{C}K_s + \cos\phi\,\vec{C}KT]\,r\ln\eta} + O(r\eta)$$

$$-3r(\vec{C} - 2(\frac{K}{2})^2\cos\phi\,\vec{b}\ln S^+) - \boxed{6(\frac{K}{2})^2\vec{b}\cos\phi\, r\ln\eta} + O(r\eta)\}$$

$$+O(r^2)$$

$$+\{2\frac{\vec{\phi}}{r} - \boxed{\vec{\phi}\frac{r}{\eta^2}} + \frac{3}{4}\vec{\phi}\frac{r^3}{\eta^4} + O(\frac{r^5}{\eta^6})\}$$

$$+\{K\cos\phi\,\vec{\phi} - \frac{3}{2}K\cos\phi\,\vec{\phi}\,(\frac{r}{\eta})^2 + O((\frac{r}{\eta})^4)$$

$$+K\vec{b}[-1+\ln 2] + \boxed{K\vec{b}\ln\eta} - K\vec{b}\ln r + \frac{3}{4}K\vec{b}(\frac{r}{\eta})^2 + O((\frac{r}{\eta})^4)\}$$

$$+\{r[(-\frac{3}{4}K^2\vec{\phi} + \frac{3}{2}K^2\cos\phi\,\vec{b})(-\frac{4}{3}+\ln 2) - \frac{1}{3}K^2\vec{\phi} + \frac{3}{4}K^2\cos^2\phi\,\vec{\phi}$$

$$+2(KT\cos\phi - K_s\sin\phi)\frac{\vec{C}}{2}(-1+\ln 2)]$$

$$+\boxed{(-\frac{3}{4}K^2\vec{\phi} + \frac{3}{2}K^2\cos\phi\,\vec{b})\,r\ln\eta} - (-\frac{3K^2}{4}\vec{\phi} + \frac{3K^2}{2}\cos\phi\,\vec{b})\,r\ln r + O((\frac{r^3}{\eta^2}))$$

$$+\boxed{(-KT\cos\phi + K_s\sin\phi)\vec{C}\,r\ln\eta} - (-KT\cos\phi + K_s\sin\phi)\vec{C}\,r\ln r\}$$

$$+O(r^2).$$

On a $\vec{\pi}\sin 2\phi + \vec{\phi}\cos 2\phi = 2\vec{\pi}\sin\phi\cos\phi + 2\vec{\phi}\cos^2\phi - \vec{\phi}$ car $\cos 2\alpha = 2\cos^2\alpha - 1$

$$= 2\cos\phi(\sin\phi\,\vec{\pi} + \cos\phi\,\vec{\phi}) - \vec{\phi}$$

$$= 2\cos\phi\,\vec{b} - \vec{\phi}.$$

$$-\frac{3}{4}K^2\vec{\phi} + \frac{3K^2}{2}\cos\phi\,\vec{b} = \frac{3}{4}K^2[2\cos\phi\,\vec{b} - \phi] = \frac{3}{4}K^2[\vec{\pi}\sin 2\phi + \vec{\phi}\cos 2\phi] = \delta$$

simplifions le terme en facteur de r :

$$\vec{B} - \frac{\vec{\Phi}}{S+2} - \frac{3}{4}\kappa^2 \ln S^+ \vec{\Phi} + \sin\phi\, \vec{C}\kappa_s \ln S^+ - \cos\phi\, \vec{C}\kappa T \ln S^+$$

$$-3\vec{C} + \frac{3}{2}\kappa^2 \vec{b} \cos\phi \ln S^+$$

$$+ g\left(-\frac{4}{3} + \ln 2\right) + \frac{3}{4}\kappa^2 \cos^2\phi\, \vec{\Phi} + \left[\kappa T\cos\phi - \kappa_s \sin\phi\right]\overset{(-2)}{\frac{\vec{C}}{2}}(-1+\ln 2) - \frac{1}{3}\kappa^2\vec{\Phi}$$

$$= \vec{B} - \frac{\vec{\Phi}}{S+2} - 3\vec{C} + g\left(-\frac{4}{3}\right) + \ln S^+\left(-\frac{3}{4}\kappa^2\vec{\Phi} + \frac{3}{2}\kappa^2\vec{b}\cos\phi\right) + g\ln 2 + \frac{3}{4}\kappa^2\left[\frac{1+\cos 2\phi}{2}\right]\vec{\Phi}$$

$$-\frac{1}{3}\kappa^2\vec{\Phi} + \vec{C}\left[(\kappa_s \sin\phi - \kappa T\cos\phi)\ln S^+ + \overset{-2}{(\kappa_s\sin\phi - \kappa T\cos\phi)}\left(1 - \frac{\ln 2}{2}\right)\right]$$

car $\cos^2 x = \frac{1+\cos 2x}{2}$

$$= \vec{B} - 3\vec{C} + g\left[\ln 2S^+ - \frac{4}{3}\right] + \frac{3}{4}\kappa^2\left[\frac{1+\cos 2\phi}{2}\right]\vec{\Phi} - \frac{\vec{\Phi}}{S+2} - \frac{1}{3}\kappa^2\vec{\Phi}$$

$$+ \vec{C}\left[\kappa_s\sin\phi - \kappa T\cos\phi\right]\left[\ln S^+ + \overset{(-2)}{1 - \frac{\ln 2}{2}}\right] \quad \frac{1}{(-2)}$$

$$= \vec{B} - 3\vec{C} + g\left[\ln 2S^+ - \frac{4}{3}\right] + \frac{3}{8}\kappa^2\cos 2\phi\,\vec{\Phi} + \kappa^2\vec{\Phi}\left(\frac{3}{8} - \frac{1}{3}\right) - \frac{\vec{\Phi}}{S+2}$$

$$+ \vec{C}\left[\kappa_s\sin\phi - \kappa T\cos\phi\right]\left[\ln S^+ \cancel{+ 1 - \frac{\ln 2}{2}}\right]$$
$$-1 + \ln 2$$

D'où :

$$\boxed{\begin{aligned} \vec{v_e} = {} & \frac{\Gamma}{2\pi r}\vec{\Phi} + \frac{\Gamma\kappa}{4\pi}\left[\ln\frac{2S^+}{r} - 1\right]\vec{b} + \frac{\Gamma\kappa}{4\pi}\cos\phi\,\vec{\Phi} + \vec{A} \\ & + \frac{3}{16}\frac{\Gamma\kappa^2 r}{\pi}\left\{(\vec{r}\sin 2\phi + \vec{\Phi}\cos 2\phi)\left[\ln\frac{2S^+}{r} - \frac{4}{3}\right] + \frac{1}{2}\cos 2\phi\,\vec{\Phi} + \frac{1}{18}\vec{\Phi} + \left(\vec{B} - 3\vec{C} - \frac{\vec{\Phi}}{S+2}\right)\frac{4}{3}\right\} \\ & + \frac{\Gamma}{4\pi}\vec{C}\, r\left[\kappa_s\sin\phi - \kappa T\cos\phi\right]\left[\ln\frac{2S^+}{r} + \cancel{1 - \frac{\ln 2}{2}}\ -1\right] \end{aligned}}$$

(Voir FUKUMOTO 91)