Tourbillons
Champ de vorticité

Plan

Biblio :

J. Bousquet Méthode des singularités (Chapitre 2)

Batchelor Introduction to fluid mechanics

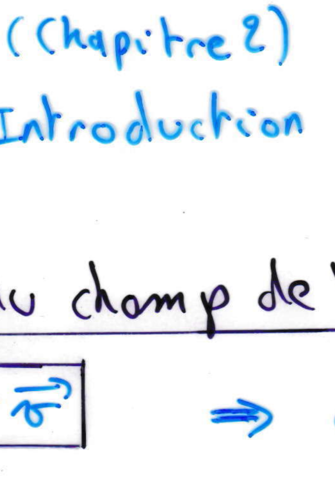
I Les équations du champ de vorticité :

Def. $\boxed{\vec{\omega} = \text{rot}\,\vec{v}}$ $\Rightarrow$ $\text{div}\,\vec{\omega} = 0$

En 3D : $\dfrac{\partial \vec{\omega}}{\partial t} + \text{Rot}(\vec{\omega} \wedge \vec{v}) = \nu \Delta \vec{\omega}$

$\Leftrightarrow \dfrac{d\vec{\omega}}{dt} = \dfrac{\partial \vec{\omega}}{\partial t} + \vec{v}.\text{grad}\,\vec{\omega} = \nu \Delta \vec{\omega} + \vec{\omega}.\text{grad}\,\vec{v}$

En 2D

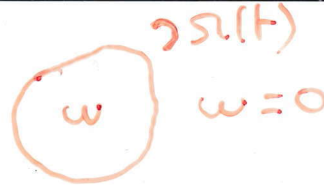


$\vec{\omega}.\text{grad}\,\vec{v} = 0$

$\dfrac{d\vec{\omega}}{dt} = \nu \Delta \vec{\omega}$

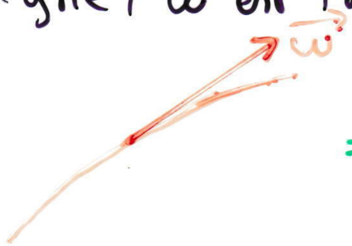
$\nu = 0 \Rightarrow \vec{\omega}$ se déplace mais garde la même intensité

$t=0$



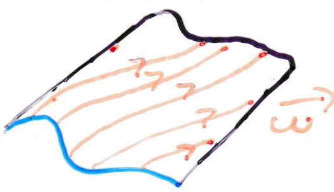
t

ligne tourbillon : ligne / $\vec{\omega}$ est tangent en tout point

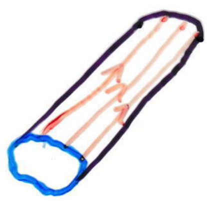


$\nu = 0$
$\Rightarrow$ ligne gelée dans le fluide

Surface tourbillon : surface engendrée par les lignes tourbillons s'appuyant sur une courbe donnée.



tube tourbillon : surface tourbillon s'appuyant sur une courbe fermée.



$$\Gamma = \int_C \vec{v}.\vec{dl} = \iint \vec{\omega}.\vec{dS}$$

$\Gamma =$ cst le long du tube.

$\nu = 0 \Rightarrow \frac{d\Gamma}{dt} = 0$ (Kelvin).

Remarque :
- écoulement irrotationnel
- domaine de vorticité faible.

Notion de vitesse induite :

- $$\boxed{\begin{aligned} \operatorname{div} \vec{v} &= 0 \\ \operatorname{rot} \vec{v} &= \vec{\omega} \end{aligned}}$$

$\Longleftrightarrow \quad \vec{v}(\underset{\sim}{x}) = \operatorname{grad} \varphi + \vec{v}_{\text{induit}}$

avec $\quad \vec{v}_{\text{induit}}(\underset{\sim}{x}) = \dfrac{1}{4\pi} \displaystyle\int_{\Omega} \dfrac{\vec{\omega}(\underset{\sim}{\xi}) \wedge (\underset{\sim}{x} - \underset{\sim}{\xi})}{|\underset{\sim}{x} - \underset{\sim}{\xi}|^3} \, d\underset{\sim}{\xi}$

- $$\boxed{\begin{aligned} \operatorname{div} \vec{v} &= \beta \\ \operatorname{rot} \vec{v} &= 0 \end{aligned}}$$

$\Longleftrightarrow \quad \vec{v}_{\text{induit}}(\underset{\sim}{x}) = \dfrac{1}{4\pi} \displaystyle\int_{\Omega} \beta(\underset{\sim}{\xi}) \, \dfrac{(\underset{\sim}{x} - \underset{\sim}{\xi})}{|\underset{\sim}{x} - \underset{\sim}{\xi}|^3} \, d\underset{\sim}{\xi}$

<u>Remarque</u> : $\quad \displaystyle\iiint_{D} \operatorname{div} \vec{v} \, d\underset{\sim}{x} = \int_{\partial D} \vec{v} \cdot \vec{n} \, ds = \text{débit}$

II Champs de vorticité concentrés :

1) Rappels sur les distributions :

11 Définition

* Champ scalair

$\mathcal{D} = \{ \varphi \, / \, \infty$ dérivable de $\mathbb{R}^n \to \mathbb{R}$ et à support compact $\}$

T = forme linéaire continue sur $\mathcal{D}$.

$$T : \begin{array}{l} \mathcal{D} \longrightarrow \mathbb{R} \\ \varphi \longmapsto \langle T, \varphi \rangle \end{array}$$

Distribution T_f associée à une fonction f :

$$\langle T_f, \varphi \rangle = \iiint f \varphi \, d\underset{\sim}{x}$$

* <u>Champ vectoriel</u> :

$\mathcal{D}$: pareil

$\vec{T}$ = forme linéaire continue de $\mathcal{D}$ sur $\mathbb{R}^n$

$$\vec{T} : \begin{array}{l} \mathcal{D} \longrightarrow \mathbb{R}^n \\ \varphi \longmapsto \langle \vec{T}, \varphi \rangle \end{array}$$

Distribution $\vec{T}_{\vec{f}}$ associée à une fonction $\vec{f}$:

$$\langle \vec{T}_{\vec{f}}, \varphi \rangle = \iiint \vec{f} \varphi \, d\underset{\sim}{x}$$

* <u>Intérêt</u> :

$$\lim_{i \to \infty} T_{f_i} = T \qquad \varepsilon = \frac{1}{i}$$

T n'est pas une fonction

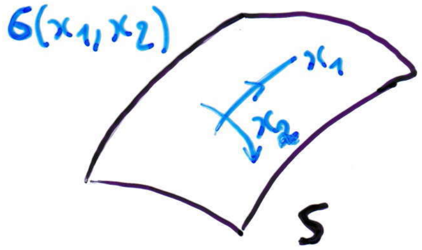
T est le <u>premier ordre</u> d'un développement en ε

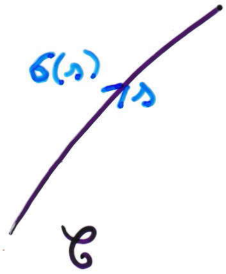
$\langle T, \varphi \rangle$ = action de T sur φ ↳ model simple

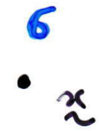
Cette action est définie.

2) Champs scalairs concentrés :

$\mathbb{R}^3$ non concentré : $\sigma(\underset{\sim}{x})$ champ volumique







Distribution surfacique — linéique — ponctuelle

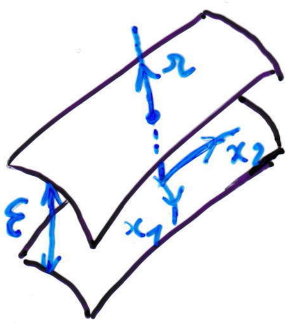
$\sigma\,\delta_S$ — $\sigma\,\delta_{\mathcal{C}}$ — $\sigma\,\delta_{\underset{\sim}{x}}$

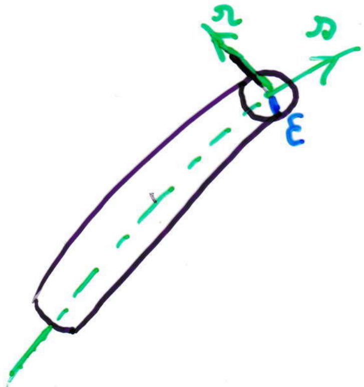
$\langle \sigma\delta_S, \varphi\rangle = \iint_S \sigma\varphi(x_1,x_2)\,dS$ — $\langle \sigma\delta_{\mathcal{C}}, \varphi\rangle = \int_{\mathcal{C}} \sigma\varphi(s)\,ds$ — $\langle \sigma\delta_{\underset{\sim}{x}}, \varphi\rangle = \sigma\varphi(\underset{\sim}{x})$

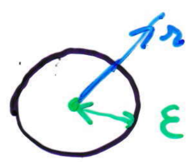
__Remarque__ : action d'un champ volumique $\sigma(\underset{\sim}{x})$

$$\langle T_{\sigma(\underset{\sim}{x})}, \varphi\rangle = \iiint \sigma(\underset{\sim}{x})\,\varphi\,d\underset{\sim}{x}$$

Les champs concentrés δ_S, $\delta_{\mathcal{C}}$ et $\delta_{\underset{\sim}{x}}$ sont les limites d'une suite de champs volumiques. f_i $\quad \varepsilon = \frac{1}{i}$



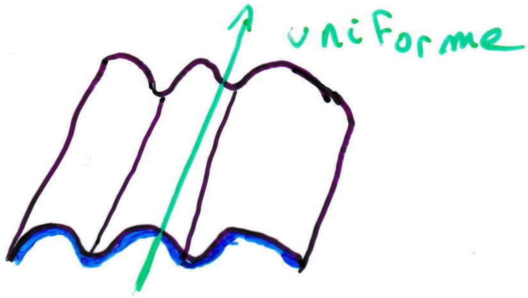


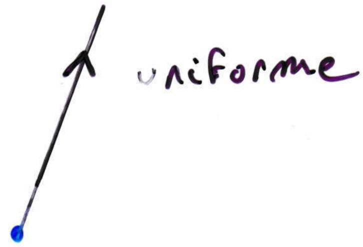


$f_i = \frac{1}{\varepsilon}$ si $r \in [0,\varepsilon]$
$= 0$ sinon

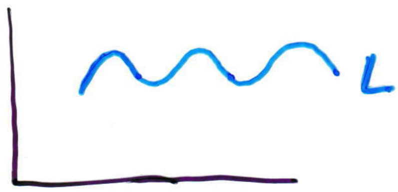
$f_i = \frac{1}{\pi\varepsilon^2}$ si $r \in [0,\varepsilon]$
$= 0$ sinon

$f_i = \frac{1}{\frac{4}{3}\pi\varepsilon^3}$ si $r \in [0,\varepsilon]$
$= 0$ sinon

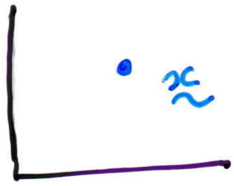




$\Downarrow$ 2D



δ_L



$\delta_{\underset{\sim}{x}}$

Exemples :

- densité de charge électrique :
  charge surfacique, linéique, électrique
- débit volumique :
  source surfacique, linéique, ponctuelle

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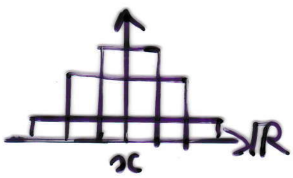
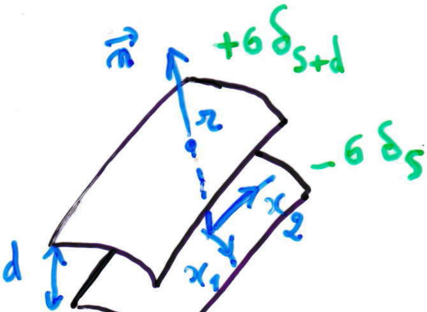
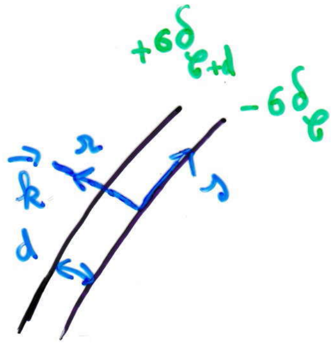
Distribution de $\mathbb{R}$:

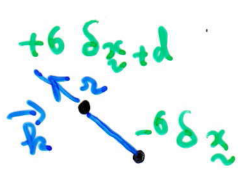


δ_x

Dérivées de distribution concentrée :







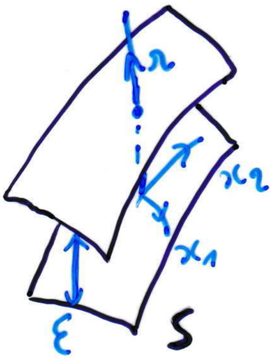
$\sigma d = K$

$\lim\limits_{d \to 0}$ doublet surfacique — doublet linéique — doublet ponctuel

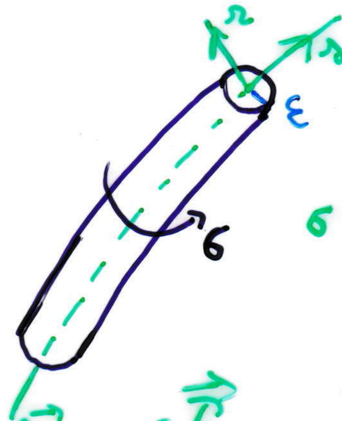
$-K \dfrac{\partial \delta_S}{\partial \vec{n}}$ $\qquad -K \dfrac{\partial \delta_\ell}{\partial \vec{k}}$ $\qquad -K \dfrac{\partial \delta_{\underset{\sim}{x}}}{\partial \vec{k}}$

3) Champ vectoriel concentré : vorticité.

$\varepsilon = \frac{1}{i}$



$\Gamma\,\delta_S\,\vec{\tau}$



$\Gamma\,\delta_\ell\,\vec{\tau}$

$\vec{f}_i = \frac{\Gamma}{\varepsilon}\,\vec{\tau}(x_1,x_2)$ si $n \in [0,\varepsilon]$

$= 0$ sinon

nappe de vorticité

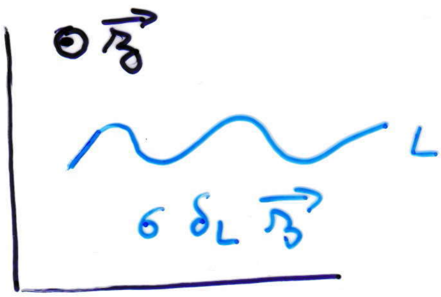
$\vec{f}_i = \frac{\Gamma\,\vec{\tau}}{\pi\varepsilon^2}$ si $r \in [0,\varepsilon]$

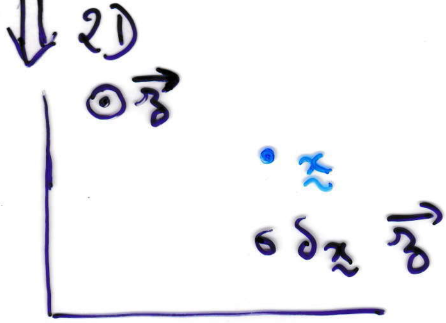
$= 0$ sinon

Γ = circulation

$\Gamma(r) = \Gamma$ + Kelvin

$\Downarrow$ 2D

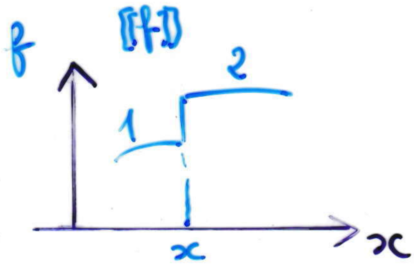




point tourbillon.

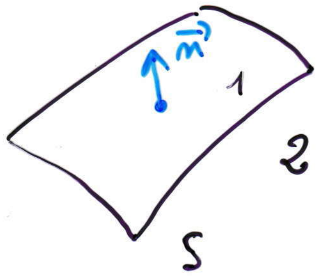
4) Discontinuités

* **Dans $\mathbb{R}$:**



$$\frac{dTf}{dx} = T\frac{df}{dx} + [\![f]\!]\,\delta_x$$

* Dans $\mathbb{R}^3$:

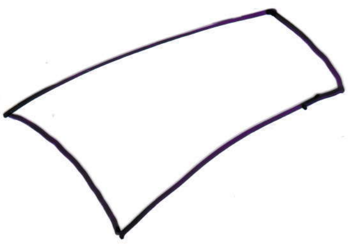


$[\![\vec{f}]\!] = \vec{f_1} - \vec{f_2}$

$$\operatorname{div} T_{\vec{f}} = T_{\operatorname{div}\vec{f}} + [\![\vec{f}]\!].\vec{n}\,\delta_S$$

$$\operatorname{rot} T_{\vec{f}} = T_{\operatorname{rot}\vec{f}} + \vec{n} \wedge [\![\vec{f}]\!]\,\delta_S$$

Exemple : champ de vitesse



$[\![\vec{v}]\!] = \vec{V_1} - \vec{V_2} = \lambda\vec{n} + \vec{\tau}$

$$\operatorname{div} \vec{v} = \lambda\,\delta_S$$
$$\operatorname{rot} \vec{v} = (\vec{\tau} \wedge \vec{n})\,\delta_S$$

5) Solutions élémentaires :

$$\left.\begin{array}{l} \Delta\varphi = \delta_0 \\ \operatorname{grad}\varphi \underset{\infty}{\to} 0 \end{array}\right\} \iff \varphi = -\frac{1}{4\pi r} \text{ dans } \mathbb{R}^3$$
$$\iff \varphi = \frac{1}{2\pi}\log r \text{ dans } \mathbb{R}^2$$

Or $\left.\begin{array}{l}\operatorname{div}\vec{v} = 0 \\ \operatorname{rot}\vec{v} = \vec{\omega}\end{array}\right) \Rightarrow \vec{v} = \operatorname{rot}\vec{A} \Big) \Rightarrow \Delta\vec{A} = \vec{\omega}$

$A = \vec{\omega} * \Psi \Rightarrow \Delta\Psi = \delta \Rightarrow$ Biot

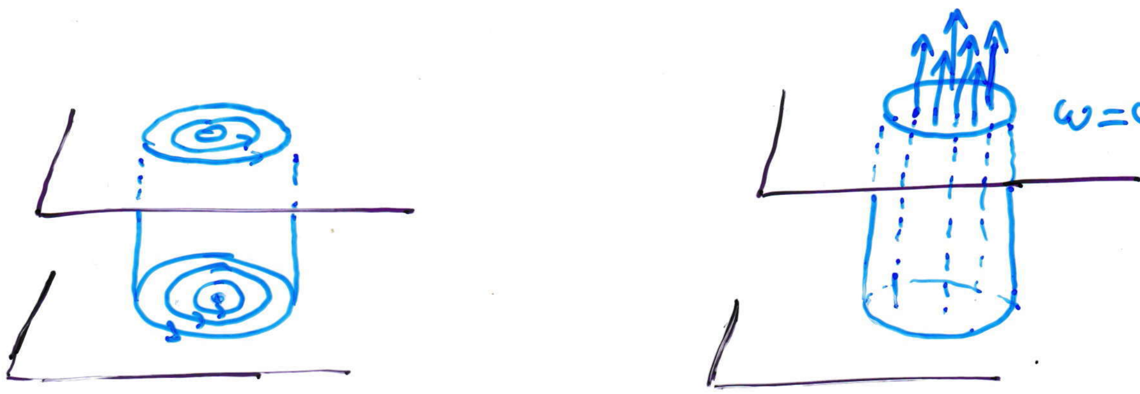
$\left.\begin{array}{l}\operatorname{div}\vec{v} = B \\ \operatorname{rot}\vec{v} = 0\end{array}\right) \Rightarrow \vec{v} = \operatorname{grad}\varphi \Big) \Rightarrow \Delta\varphi = B \Rightarrow \varphi = B * \Psi$

$\Delta\Psi = \delta$

III Exercices sur les champs induits:

1) Tornade : $\omega \Rightarrow v$

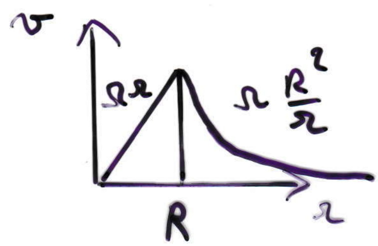
$\omega = 2\Omega = \text{cst}$



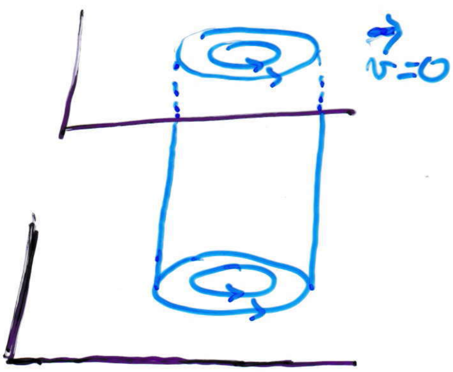
Lignes de courant

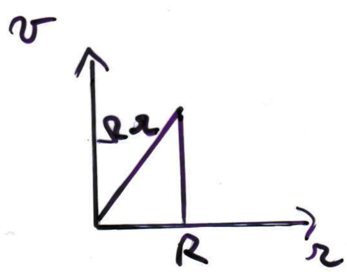
Lignes de vorticité

Stokes ou BSS 2D $\Rightarrow$



2) Colonne en rotation : $v \Rightarrow \omega$



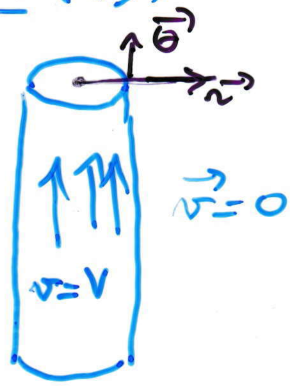


$[[\vec{v}]] = (0 - \Omega R)\vec{\theta}$

$K = -\Omega R$

$\vec{\omega} = K \delta_S \vec{z} + \text{vorticité cst}$

3) Jet : (3D)



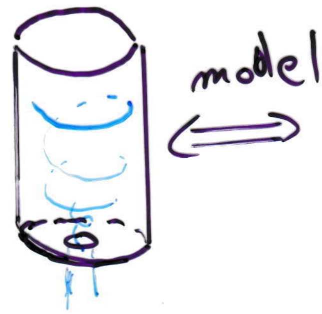
$[\![\vec v]\!] = -V\vec z$

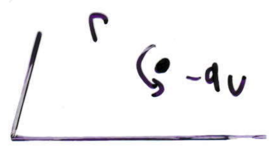
$K = V$

$\vec\omega = K\,\delta_S\,\vec\theta$

vitesse

4) Vidange



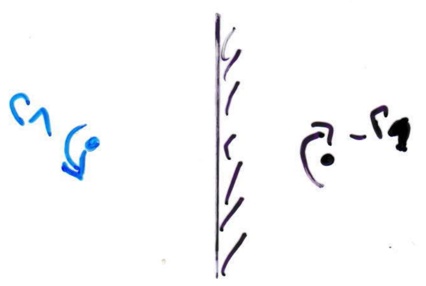


puit-tourbillon

5) 2 tourbillons:

linéaire ⇒ supperposition.

6) Théo des images:



$\vec v \cdot \vec n = 0$

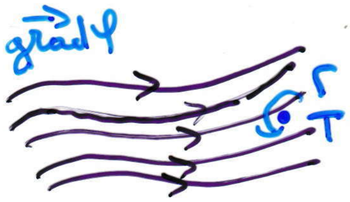
IV Instationnarité – Mouvement :

$$\left.\begin{array}{l} CI \\ CL \end{array}\right\} \Rightarrow \begin{array}{l} \vec{\omega}(t) \\ B(t) \end{array}$$

Utilisation des équations dynamiques.

En 2D et $\nu = 0$: lignes tourbillons sont gelées.

1) Tourbillon dans un écoulement :



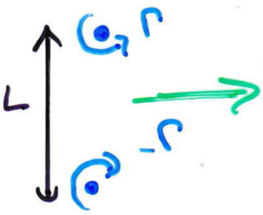
$$\vec{v}(\underline{x}) = \vec{\text{grad}}\,\varphi + \vec{v}_{induit}$$

vorticité gelée $\Rightarrow$ $\vec{v}(T) = \vec{\text{grad}}\,\varphi\,(T)$

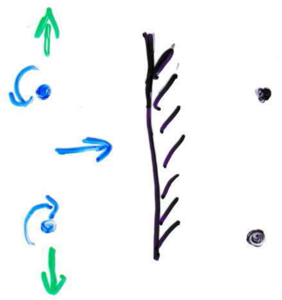
Remarque : $\vec{v}_{induit}(T) = \odot = \dfrac{1}{2\pi r}\vec{e}_\theta$

moyenne nulle.

2) Deux points tourbillons :

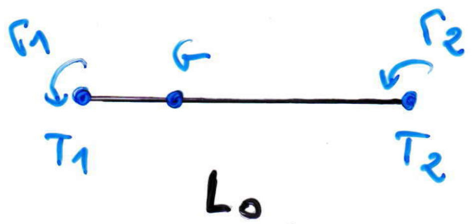


$$\vec{v} = \frac{\Gamma}{2\pi L}$$



cuillère

génération vorticité : mouvement | paroi
| surface libre.



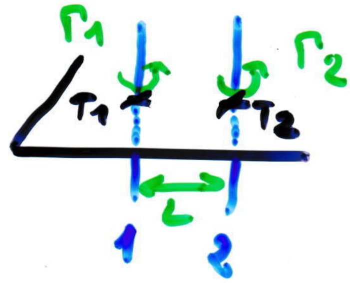
$$\Gamma_1 \, \overrightarrow{GT_1} + \Gamma_2 \, \overrightarrow{GT_2} = 0$$

G immobile

$L_0 = \text{cst}$

rotation $\Omega = \dfrac{\Gamma_1 + \Gamma_2}{2\pi L_0^2}$

Exemple : les deux filets tourbillons

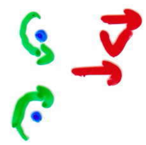


G : centre de vorticité :

$$\Gamma_1 \overrightarrow{GT_1} + \Gamma_2 \overrightarrow{GT_2} = \vec{0}$$

G est invariant.

- Si $\Gamma_1 + \Gamma_2 = 0$:



Il y a translation à la vitesse $V = \dfrac{\Gamma_1}{2\pi L}$.

- Si $\Gamma_1 + \Gamma_2 \neq 0$:

Il y a rotation autour de G à la vitesse

$$\Omega = \frac{\Gamma_1 + \Gamma_2}{2\pi L^2}.$$

Résolution :

$T_1(x_1, y_1)$ $T_2(x_2, y_2)$

tourbillon $\rightarrow$ $\psi = -\dfrac{\Gamma}{2\pi} \log(r)$

D'où : $\psi_1(x,y) = -\dfrac{1}{4\pi} \Gamma_2 \log\left[(x-x_2)^2 + (y-y_2)^2\right]$

$\psi_2(x,y) = -\dfrac{1}{4\pi} \Gamma_1 \log\left[(x-x_1)^2 + (y-y_1)^2\right]$

$$\left|\begin{array}{l} \dfrac{dx_1}{dt} = \dfrac{\partial \psi_1}{\partial y} \\[2ex] \dfrac{dy_1}{dt} = -\dfrac{\partial \psi_1}{\partial x} \end{array}\right. \qquad \left|\begin{array}{l} \dfrac{dx_2}{dt} = \dfrac{\partial \psi_2}{\partial y} \\[2ex] \dfrac{dy_2}{dt} = -\dfrac{\partial \psi_2}{\partial x} \end{array}\right.$$

Soit $r^2 = (x_1-x_2)^2 + (y_1-y_2)^2 = L^2$

$$\frac{dr^2}{dt} = \left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)(x_1-x_2) + \left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right)(y_1-y_2)$$

Or : $\frac{dx_1}{dt} = -\frac{1}{4\pi}\frac{\Gamma_2}{L^2} 2(y_1-y_2)$ $\quad \frac{dy_1}{dt} = \frac{1}{4\pi}\Gamma_2 2(x_1-x_2)$

$\frac{dx_2}{dt} = -\frac{1}{4\pi}\frac{\Gamma_1 2}{L^2}(y_1-y_2)$ $\quad \frac{dy_2}{dt} = \frac{1}{4\pi}\Gamma_1 2(x_1-x_2)$

On a donc $\frac{dr^2}{dt} = 0$ d'où $r = L = cst.$

- Si $\underline{\Gamma_1+\Gamma_2 = 0}$: on constate que :

$$\frac{dx_1}{dt} = \frac{dx_2}{dt} \text{ et } \frac{dy_1}{dt} = \frac{dy_2}{dt}.$$

il y a translation à une vitesse $\vec{v}$

Or l'expression de la vitesse complexe induit par un tourbillon ponctuel est $u - iv = \frac{-i\Gamma}{2\pi z}$ d'où $V = \frac{\Gamma_1}{2\pi L}$

- Si $\underline{\Gamma_1+\Gamma_2 \neq 0}$:

G vérifie $\Gamma_1 \vec{GT_1} + \Gamma_2 \vec{GT_2} = \vec{0}$

d'où $(\Gamma_1+\Gamma_2)\vec{GT_1} + \Gamma_2 \vec{T_1T_2} = 0$

$$\vec{T_2T_1} = \frac{\Gamma_1+\Gamma_2}{\Gamma_2}\vec{GT_1}$$

G invariant

↳ G : centre du repère

$$\frac{dx_1}{dt} = -\frac{1}{4\pi}\frac{\Gamma_2}{L^2}2(y_1-y_2) = -\frac{1}{2\pi}\frac{\Gamma_1+\Gamma_2}{L^2}y_1$$

$$\frac{dy_1}{dt} = \frac{1}{4\pi}\frac{\Gamma_2}{L^2}2(x_1-x_2) = \frac{1}{2\pi}\frac{\Gamma_1+\Gamma_2}{L^2}x_1.$$

Il y a rotation autour de G à la vitesse :

$$\Omega = \frac{\Gamma_1+\Gamma_2}{2\pi L^2}.$$

Exemple : invariance du centre de vorticité. (2D).

$$X_i = \frac{\int_{\mathbb{R}^2} x_i \omega \, d\underset{\sim}{x}}{\int_{\mathbb{R}^2} \omega \, d\underset{\sim}{x}}$$ centre de vorticité

En 2D $\frac{d\vec{\omega}}{dt} = v_{i,j}\omega_j$ devient $\frac{d\omega}{dt} = 0$.

D'où
$$\frac{d}{dt}\int_{\mathbb{R}^2} \omega \, d\underset{\sim}{x} = \int_{\mathbb{R}^2} \frac{d\omega}{dt} d\underset{\sim}{x} = 0.$$

Montrons que l'on a aussi $\frac{d}{dt}\int_{\mathbb{R}^2} x_i \omega \, d\underset{\sim}{x} = 0.$

$$\frac{d}{dt}\int_{\mathbb{R}^2} x_i \omega \, d\underset{\sim}{x} = \int_{\mathbb{R}^2} v_i \omega \, d\underset{\sim}{x}$$

Notons. $v_1 = u$ et $v_2 = v$

On remarque que :

$$\frac{\partial(u^2-v^2)}{\partial y} - 2\frac{\partial(uv)}{\partial x} = 2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} - 2u\frac{\partial v}{\partial x} - 2v\frac{\partial u}{\partial x}$$
$$= -2v\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - 2u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
$$= -2u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \text{ car } \operatorname{div}\vec{v} = 0.$$
$$= -2\omega u.$$